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BASIC DOUBLE HYPERGEOMETRIC FUNCTIONS (II)

By F. H. JACKSON (*Eastbourne*)

[Received 6 October 1943]

1. Introduction

THIS paper is a continuation of a previous one by the writer and investigates the confluent forms of the functions. As in the first paper the expansions are obtained by using certain functions of symbolic operators, formed from basic hypergeometric series, in which two of the parameters are replaced by symbolic partial differential operators $x\partial/\partial x$, $y\partial/\partial y$. This method originated in papers* by Professor Burchnall and Mr. Chaundy, who suggested to me that it might well be adapted to basic functions and operators. It is obvious that the method can be used to obtain results of great generality in basic series. I give one example in the case of a function with nine elements (a, b, c, a', b', c', h, k, λ). It is, however, in the simple special cases connected with the addition theorems of basic Bessel, Laguerre, and other functions that results of most interest may be found. The late Professor Forsyth once suggested to me that, if the base q in the operator $(1-q^{\theta})/x(1-q)$ and also in the basic functions were replaced by $1+\epsilon$, the resulting analysis might possibly prove useful in dealing with physical problems in which assumed conditions are never in exact accord with reality, whatever that may be. I hope to develop this suggestion.

In this paper I make a change in the notation from that of my first paper, by introducing a factor λ into the index of the solitary factors so that $q^{\lambda n(n-1)}$ replaces $q^{n(n-1)/2}$. This change does away with the distinction between normal and abnormal functions. It permits also of the use of the following lemma:

LEMMA.

$$\sum_{m+n=N} \frac{q^{nc'+n(n-1)}}{(m)!\,(n)!\,(c)_m(c')_n} = \frac{[c+c']_N[c+c'-1]_N}{(N)!\,(c)_N(c')_N(c+c'-1)_N}$$

in which

$$(c)_N = (1-q^c)(1-q^{c+1})...(1-q^{c+N-1}),$$

$$[c+c']_N = (1-q^{c+c'})(1-q^{c+c'+2})...(1-q^{c+c'+2N-2}).$$

The brackets ( ), [ ] are convenient to show advances in factors by q, q^2 respectively.

* Burchnall and Chaundy, (2), (3).

This lemma enables us to effect transformations of functions denoted by $\Upsilon_2 \times \Psi_2$ and also a basic analogue of the curious $F(a, b; c; x+y-xy)$ discussed by Burchnall and Chaundy. Υ_2, Ψ_2 are connected with the basic analogues of Bessel, Whittaker, and Laguerre Functions. The basic analogues of Bessel and Legendre Functions were investigated many years ago by the present writer using another method.*

2. Definitions of the functions

In the confluent functions the factors are basic numbers: thus $(a) \equiv (1-q^a)/(1-q)$.

I define the following basic functions:

$$\Upsilon_1(a; b; c; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)}, \tag{1}$$

$$\Psi_1(a; b; c, c'; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n-1)}, \tag{2}$$

$$\Xi_1(a, a'; b; c; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)}, \tag{3}$$

$$\Phi(a, b; c; x)\,{}_1\Phi_1(a'; c'; y, \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a')_n}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n-1)}, \tag{4}$$

$$\Upsilon_2(a, a'; c; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)}, \tag{5}$$

$$\Psi_2(a; c, c'; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n-1)}, \tag{6}$$

$${}_1\Phi_1(a; c; x)\,{}_1\Phi_1(a'; c'; y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a')_n}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n-1)}, \tag{7}$$

$$\Xi_2(a; b; c; x, y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(b)_m}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)}, \tag{8}$$

$$\Phi(a; b; c; x)\,{}_0\Phi_1(c'; y; \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(b)_m}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n+1)}, \tag{9}$$

$$\Upsilon_3(a; c; x, y, \lambda) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)}. \tag{10}$$

* F. H. Jackson, (4) 193, (5) 1.

When $\lambda = \frac{1}{2}$ we have, in the notation of my previous paper,

$$\Phi(a,b;c;x|+y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\frac{1}{2}n(n-1)} \tag{11}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_r}(x+y)(x+qy)...(x+q^{r-1}y), \tag{12}$$

which is the *natural* analogue of $F(a,b;c;x+y)$.

3. Elementary operations

For convenience of reference I restate the fundamental operations

$$(-\theta)_r(-\phi)_r \frac{x^m y^n}{(m)!\,(n)!} = \frac{(-m)_r(-n)_r}{(m)!\,(n)!} x^m y^n = \frac{x^m y^n q^{r(-m-n+r-1)}}{(m-r)!\,(n-r)!}, \tag{13}$$

$$(1-h-\theta-\phi)_r x^m y^n = (-)^r(h+m+n-r)_r x^m y^n q^{r(r+1)/2-r(h+m+n)}, \tag{14}$$

$$\{G_q(h)\}^{-1} = \lim_{n\to\infty}(1-q^h)(1-q^{h+1})...(1-q^{h+n})/(1-q)^n,$$

$$\Delta_q(h) = \frac{G(h+\theta)G(h+\phi)}{G(h)G(h+\theta+\phi)}, \qquad \nabla_q(h) \text{ is the inverse of } \Delta_q(h),$$

$$\nabla_q(h)x^m y^n = \frac{(h)_{m+n}}{(h)_m(h)_n} x^m y^n, \qquad \Delta_q(h)x^m y^n = \frac{(h)_m(h)_n}{(h)_{m+n}} x^m y^n. \tag{15}$$

Applying the Δ, ∇ operators to the functions (1),..., (12) we obtain

$$\Upsilon_1(a;b;c;x,y;\lambda) = \nabla_q(a)\Delta_q(c)\Phi(a,b;c;x)\,{}_1\Phi_1(a;c;y;\lambda) \tag{16}$$

$$= \nabla_q(a)\Xi_1(a,a;b;c;x,y;\lambda) \tag{17}$$

$$= \Delta_q(c)\Psi_1(a;b;c,c;x,y;\lambda), \tag{18}$$

$$\Xi_1(a,a',b;c;x,y;\lambda) = \Delta_q(c)\Phi(a,b;c;x)\,{}_1\Phi_1(a';c;y;\lambda), \tag{19}$$

$$\Psi_1(a;b;c,c';x,y;\lambda) = \nabla_q(a)\Phi(a,b;c;x)\,{}_1\Phi_1(a;c';y;\lambda), \tag{20}$$

$$\Xi_2(a;b;c;x,y;\lambda) = \Delta_q(c)\Phi(a,b;c;x)\,{}_0\Phi_1(c;y;\lambda), \tag{21}$$

$$\Upsilon_3(a;c;x,y;\lambda) = \Delta_q(c)\,{}_1\Phi_1(a;c;x)\,{}_0\Phi_1(c;y;\lambda), \tag{22}$$

$${}_1\Phi_1(a;c;x|+y) = \nabla_q(a)\Delta_q(c)\,{}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a;c;y;\tfrac{1}{2}) \tag{23}$$

$$= \nabla_q(a)\Upsilon_2(a,a;c;x,y;\tfrac{1}{2}) \tag{24}$$

$$= \Delta_q(c)\Psi_2(a;c,c;x,y;\tfrac{1}{2}), \tag{25}$$

$${}_0\Phi_1(c;x|+y) = \Delta_q(c)\,{}_0\Phi_1(c;x)\,{}_0\Phi_1(c;y;\tfrac{1}{2}), \tag{26}$$

$$\Upsilon_2(a,a';c;x,y;\lambda) = \Delta_q(c)\,{}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a';c;y;\lambda), \tag{27}$$

$$\Psi_2(a;c,c';x,y;\lambda) = \nabla_q(a)\,{}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a;c';y;\lambda). \tag{28}$$

For each of the operators $\nabla, \Delta, \nabla\Delta$; there is an equivalent hypergeometric operator series. It is convenient to divide the expansions into five groups, each group under its appropriate operator series. Detailed analysis is given in one case only: special care is needed for right determination of the q-solitary factors.

3.1. *The operator function*

$$\nabla_q(h) = \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(h)_r} q^{r(h+\theta+\phi)}. \tag{29}$$

We can obtain eleven expansions by applying this operator to the identities (17), (20), (24), (28), and to the inverses of (17), (19), (21), (22), (25), (26), (27). As an example, consider its effect on Ψ_1 as given by (20). Writing a for h in (29) we have

$$\begin{aligned}
&\Psi_1(a,b;c,c';x,y;\lambda)\\
&= \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(a)_r} q^{r(a+\theta+\phi)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a)_n}{(m)!\,(n)!\,(c)_m(c')_n} x^m y^n q^{\lambda n(n-1)}\\
&= \sum_{r=0}^{\infty}\sum_{m=r}^{\infty}\sum_{n=r}^{\infty} \frac{(-m)_r(-n)_r(a)_m(b)_m(a)_n}{(m)!\,(n)!\,(r)!\,(a)_r(c)_r(c')_r} q^{r(a+m+n)+\lambda n(n-1)} x^m y^n,\\
&\qquad \text{since } (-m)_r(-n)_r \equiv 0 \text{ for } \min(m,n) < r,\\
&= \sum_{r=0}^{\infty}\sum_{m=r}^{\infty}\sum_{n=r}^{\infty} \frac{(a)_m(b)_m(a)_n}{(m-r)!\,(n-r)!\,(a)_r(c)_r(c')_r} \times\\
&\qquad\qquad \times q^{r(a+m+n)+\lambda n(n-1)+r(-m-n+r-1)} x^m y^n.
\end{aligned}$$

We replace m, n by $M+r$, $N+r$, $(a)_{M+r}$ by $(a)_r(a+r)_M$, and so on. This gives

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_r(c')_r} x^r y^r q^{ra+(1+\lambda)r(r-1)} \times\\
&\qquad \times \sum_{M=0}^{\infty}\sum_{N=0}^{\infty} \frac{(a+r)_M(b+r)_M(a+r)_N}{(M)!\,(N)!\,(c+r)_M(c'+r)_N} q^{2\lambda rN+\lambda N(N-1)} x^M y^N\\
&= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_r(c')_r} x^r y^r q^{ra+(1+\lambda)r(r-1)} \times\\
&\qquad \times \Phi(a+r;b+r;c+r;x)\,{}_1\Phi_1(a+r;c'+r;yq^{2\lambda r};\lambda).
\end{aligned} \tag{30}$$

If we put $\lambda = 0$, we have a confluent form of one of the expansions of the former paper; if $\lambda = \frac{1}{2}$, we have the so-called abnormal form. In connexion with addition theorems for basic Bessel and other

functions we have $\lambda = 1$. Inverting (22), we obtain by using the operator function $\nabla_q(c)$ the expansion

$$
{}_1\Phi_1(a;c;x)\,{}_0\Phi_1(c;y;\lambda)
$$

$$
= \sum_{r=0}^{\infty} \frac{(a)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)} \Upsilon_3(a+r;c+2r;x,yq^{2\lambda r};\lambda). \quad (31)
$$

Similarly, from (25) by inverting we get

$$
\Psi_2(a;c,c;x,y;\lambda)
$$

$$
= \sum_{r=0}^{\infty} \frac{(a)_{2r}}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)}\, {}_1\Phi_1(a+2r;c+2r;x,yq^{2\lambda r};\lambda). \quad (32)
$$

We should note that when $\lambda = \frac{1}{2}$, the function ${}_1\Phi_1$ becomes ${}_1\Phi_1(a+2r;c+2r;x|+q^r y)$.

From the other identities involving ∇_q we similarly have

$$
{}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a';c;y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(a')_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)} \times
$$

$$
\times \Upsilon_2(a+r,a'+r;c+2r;x,yq^{2\lambda r};\lambda), \quad (33)
$$

$$
\Phi(a,b;c;x)\,{}_1\Phi_1(a',c;y,\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(a')_r(b)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)} \times
$$

$$
\times \Xi_1(a+r,a'+r;b+r;c+2r;x,yq^{2r\lambda};\lambda), \quad (34)
$$

$$
\Upsilon_1(a;b;c;x,y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_{2r}} x^r y^r q^{ra+(1+\lambda)r(r-1)} \times
$$

$$
\times \Xi_1(a+r,a+r;b+r;c+2r;x,yq^{2r\lambda};\lambda), \quad (35)
$$

$$
\Psi_1(a;b;c,c;x,y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)} \times
$$

$$
\times \Upsilon(a+2r;b+r;c+2r;x,yq^{2r\lambda};\lambda), \quad (36)
$$

$$
\Phi(a,b;c;x)\,{}_0\Phi_1(c;y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(1+\lambda)r(r-1)} \times
$$

$$
\times \Xi_2(a+r;b+r;c+2r;x,q^{2r\lambda}y;\lambda), \quad (37)
$$

$$
{}_1\Phi_1\{a;c;(x|+y)\lambda\} = \sum_{r=0}^{\infty} \frac{(a)_r}{(r)!\,(c)_{2r}} x^r y^r q^{ra+(1+\lambda)r(r-1)} \times
$$

$$
\times \Upsilon_2(a+r,a+r;c+2r;x,yq^{2r\lambda};\lambda), \quad (38)
$$

$$\Psi_2(a;c,c';x,y;\lambda)=\sum_{r=0}^{\infty}\frac{(a)_r}{(r)!\,(c)_r(c)_{2r}}x^ry^rq^{ra+(1+\lambda)r(r-1)}\times$$
$$\times{}_1\Phi_1(a+r;c+r;x)\,{}_1\Phi_1(a+r;c'+r;yq^{2r\lambda};\lambda), \quad (39)$$

$${}_0\Phi_1(c;x)\,{}_0\Phi_1(c;y;\lambda)=\sum_{r=0}^{\infty}\frac{1}{(r)!\,(c)_r(c)_{2r}}x^ry^rq^{rc+(1+\lambda)r(r-1)}\times$$
$$\times{}_0\Phi_1\{c+2r;(x|+y^{2r\lambda});\lambda\}. \quad (40)$$

3.2. *The operator function*

$$\Delta_q(h)=\sum_{r=0}^{\infty}\frac{(h)_{2r}(-\theta)_r(-\phi)_r}{(r)!\,(h+r-1)_r(h+\theta)_r(h+\phi)_r}q^{r(r-1)/2+r(h+\theta+\phi)}. \quad (41)$$

From this operator we obtain the seven series

$$\Xi_1(a,a';b;c;x,y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(a')_r(b)_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+2r;x)\,{}_1\Phi_1(a'+r;c+2r;q^{2\lambda r}y;\lambda), \quad (42)$$

$$\Upsilon_1(a;b;c;x,y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_{2r}(b)_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times\Psi_1(a+2r,b+r;c+2r,c+2r;x,q^{2r\lambda}y;\lambda), \quad (43)$$

$$\Upsilon_2(a,a';c;x,y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(a')_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times{}_1\Phi_1(a+r;c+2r;x)\,{}_1\Phi_1(a'+r;c+2r;yq^{2\lambda r};\lambda), \quad (44)$$

$${}_1\Phi_1(a;c;x|+y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_{2r}}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times\Psi_2(a+2r;c+2r,c+2r;x,yq^{2r\lambda};\lambda), \quad (45)$$

$$\Xi_2(a;b;c;x,y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(b)_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times\Phi(a+r,b+r;c+2r;x)\,{}_0\Phi_1(c+2r;q^{2r\lambda}y;\lambda), \quad (46)$$

$$\Upsilon_3(a;c;x,y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r}{(r)!\,(c+r-1)_r(c)_{2r}}x^ry^rq^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times{}_1\Phi_1(a+r;c+2r;x)\,{}_0\Phi_1(c+2r;yq^{2r\lambda};\lambda), \quad (47)$$

$${}_0\Phi_1(c;x|+y;\lambda)=\sum_{r=0}^{\infty}(-)^r\frac{x^ry^r}{(r)!\,(c+r-1)_r(c)_{2r}}q^{rc+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times{}_0\Phi_1(c+2r;x)\,{}_0\Phi_1(c+2r,q^{2r\lambda}y,\lambda). \quad (48)$$

3.3. *The operator function*

$$\Delta_q(h) = \sum_{r=0}^{\infty} \frac{(-\theta)_r(-\phi)_r}{(r)!\,(1-h-\theta-\phi)_r} q^r. \tag{49}$$

This gives the four identities

$$\Phi(a,b;c;x)\,{}_1\Phi_1(a;c';y;\lambda) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r}{(r)!\,(c)_r(c')_{2r}} x^r y^r q^{ra+(\lambda+\frac{1}{2})r(r-1)} \times$$
$$\times \Psi_1(a+r;b+r;c+r,c'+r,x,yq^{2\lambda r};\lambda), \tag{50}$$

$${}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a;c';y,\lambda) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r}{(r)!\,(c)_r(c')_r} x^r y^r q^{ra+(\lambda+\frac{1}{2})r(r-1)} \times$$
$$\times \Psi_2(a+r;c+r,c'+r;x,yq^{2r\lambda};\lambda), \tag{51}$$

$$\Xi_1(a,a;b;c;x,y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_{2r}} x^r y^r q^{ra+(\lambda+\frac{1}{2})r(r-1)} \times$$
$$\times \Upsilon_1(a+r;b+r;c+2r;x,yq^{2r\lambda};\lambda), \tag{52}$$

$$\Upsilon_2(a,a;c;x,y;\lambda) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r}{(r)!\,(c)_{2r}} x^r y^r q^{ra+(\lambda+\frac{1}{2})r(r-1)} \times$$
$$\times {}_1\Phi_1(a+r;c+2r;x,yq^{2r\lambda};\lambda). \tag{53}$$

3.4. *The operator function*

$$\nabla_q'(h)\Delta_q(k) = \sum_{r=0}^{\infty} \frac{(k-h)_r(h)_{2r}(-\theta)_r(-\phi)_r}{(r)!\,(k+r-1)_r(k+\theta)_r(k+\phi)_r(h)_r} q^{r(r-1)/2+r(k+\theta+\phi)}. \tag{54}$$

Here we derive the two series

$${}_1\Phi_1(a;c;x|+y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(c-a)_r}{(r)!\,(c+r-1)_r(c)_{2r}} x^r y^r q^{rc+(\lambda+1)r(r-1)} \times$$
$$\times {}_1\Phi_1(a+r;c+2r;x)\,{}_1\Phi_1(a+r;c+2r;q^{2r\lambda}y;\lambda), \tag{55}$$

$$\Upsilon_1(a;b;c;x,y;\lambda) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r}{(r)!\,(c+r-1)_r(c)_{2r}} x^r y^r q^{rc+(\lambda+1)r(r-1)} \times$$
$$\times \Phi(a+r,b+r;c+2r;x)\,{}_1\Phi_1(a+r;c+2r;q^{2r\lambda}y,\lambda). \tag{56}$$

3.5. *The operator function*

$$\nabla_q(h)\Delta_q(k) = \sum_{r=0}^{\infty} \frac{(h-k)_r(-\theta)_r(-\phi)_r}{(r)!\,(h)_r(1-k-\theta-\phi)_r} q^r. \tag{57}$$

Here again we have two series

$$ {}_1\Phi_1(a;c;x)\,{}_1\Phi_1(a;c;y;\lambda) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(c-a)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{ra+(\lambda+1)r(r-1)} \times $$
$$ \times {}_1\Phi_1(a+r;c+2r;x|+q^{2r\lambda}y;\lambda), \quad (58) $$

$$ \Phi(a,b;c;x)\,{}_1\Phi_1(a;c;y;\lambda) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(c-a)_r}{(r)!\,(c)_r(c)_{2r}} x^r y^r q^{rc+(\lambda+1)r(r-1)} \times $$
$$ \times \Upsilon_1(a+r;b+r;c+2r;x,q^{2r\lambda}y;\lambda). \quad (59) $$

4. Special theorems

LEMMA I. $$ \sum_{m+n=N} \frac{(a)_m(a')_n}{(m)!\,(n)!} q^{na} = \frac{(a+a')_N}{(N)!}. \quad (60) $$

LEMMA II.
$$ \sum_{m+n=N} \frac{q^{nc'+n(n-1)}}{(m)!\,(n)!\,(c)_m(c')_n} = \frac{[c+c']_N[c+c'-1]_N}{(N)!\,(c)_N(c')_N(c+c'-1)_N}. \quad (61) $$

Of these Lemma I is the q-analogue of Vandermonde's theorem; Lemma II is a special case of a theorem due to the writer.* These lemmas are useful in effecting transformations of the functions Υ_2, Ψ_2 from which basic Whittaker's and Laguerre functions are derived. Consider the product $\Upsilon_2 \times \Psi_2$ in the form

$$ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n}{(m)!\,(n)!\,(c)_{m+n}} x^m y^n q^{\lambda n(n-1)} \times $$
$$ \times \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{(b)_{m+n}}{(\mu)!\,(\nu)!\,(d)_\mu(d')_\nu} \xi^\mu \eta^\nu q^{\lambda'\nu(\nu-1)}. \quad (62) $$

Replace λ, λ', y, η by 0, 1, xq^a, ξq^d respectively, and apply Lemma I to the series Υ_2, Lemma II to the series Ψ_2. We obtain

$$ \sum_{M=0}^{\infty} \frac{(a+a')_M}{(M)!\,(c)_M} x^M \times \sum_{N=0}^{\infty} \frac{(b)_N[d+d']_N[d+d'-1]_N}{(N)!\,(d)_N(d')_N(d+d'-1)_N} \xi^N $$
$$ = \Upsilon_2(a,a';c;x,xq^a)\Psi_2(b;d,d';\xi,\xi q^d;1). \quad (63) $$

If $a+a' = c$,

$$ \sum_{M=0}^{\infty} \frac{(a+a')_M}{(M)!\,(c)_M} x^M \quad \text{reduces to} \quad 1+\frac{x}{(1)}+\frac{x^2}{(2)!}+..., \quad (64) $$

* Jackson, (1) 68, § 8.4 (3).

the basic analogue of e^x, and we have, when $q = 1$,

$$e^x F\left\{\begin{matrix} b, \tfrac{1}{2}(d+d'), \tfrac{1}{2}(d+d'-1); 4\xi \\ d, d', d+d'-1 \end{matrix}\right\} = \Phi_2(a, a'; a+a'; x, x)\Psi_2(b; d, d'; \xi, \xi) \tag{65}$$

in the notation of the ordinary confluent functions.

5. Basic Laguerre polynomials

Defining the basic analogue as

$$L^{\alpha}_{n,q}(x) = \frac{G(\alpha+n+1)}{(n)!\, G(\alpha+1)}\, {}_1\Phi_1(-n; \alpha+1; x; \tfrac{1}{2}), \tag{66}$$

we obtain from (55), (58)

$$L^{\alpha}_{n,q}(x|+y) = \sum_{r=0}^{n} (-)^r \frac{(n-r)!\, (\alpha+2r)G(\alpha+r)}{(r)!\, G(\alpha+n+r+1)} x^r y^r q^{r\alpha+r(3r-1)/2} \times$$
$$\times L^{\alpha+2r}_{n-r}(x) L^{\alpha-2r}_{n-r}(yq^r, \tfrac{1}{2}), \tag{67}$$

$$L^{\alpha}_{n,q}(x) L_{n,q}(y, \tfrac{1}{2}) = \frac{G(\alpha+n+1)}{(n)!} \sum_{r=0}^{n} \frac{x^r y^r}{(r)!\, G(\alpha+r-1)} q^{3r(r-1)/2-rn} \times$$
$$\times L^{\alpha+2r}_{n-r}(x|+q^r y, \tfrac{1}{2}). \tag{68}$$

Formulae for $\quad L^{\alpha}_{m+n,q}(x), \quad L^{\alpha}_{m,q}(x) L^{\alpha}_{n,q}(x),$

analogues of those given by Erdélyi and Howell,* can easily be obtained, but it will suffice to say that they arise by giving special values to the parameters a, a', c in formula (63) and its inverse.

From (56) and Lemma I we obtain

$${}_1\Phi_1(a+a'; c; x) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (a')_r}{(r)!\, (c+r-1)_r (c)_{2r}} x^{2r} q^{rc+r(r-1)} \times$$
$$\times {}_1\Phi_1(a+r; c+2r; x)\, {}_1\Phi_1(a'+r; c+2r; x), \tag{69}$$

from which an expansion for the basic function $L^{\alpha}_{m+n,q}(x)$ can be derived, namely,

$$\frac{(m)!\, (n)!\, G(\alpha+m+n+1)}{(m+n)!} \times$$
$$\times \sum_{r=0}^{\min(m,n)} (-)^r \frac{(\alpha+2r)G(\alpha+r)x^{2r}}{G(\alpha+m+r+1)G(\alpha+n+r+1)} q^{r\alpha+r(r-1)} L^{\alpha+2r}_{m-r,q}(x) L^{\alpha+2r}_{n-r,q}(x). \tag{70}$$

* Cf. (2) 128.

6. The basic Bessel function

From (47) we have

$$ {}_0\Phi_1(c;x|+y;\tfrac{1}{2}) = \sum \frac{q^{r(r-1)/2}}{(r)!\,(c)_r}(x+y)(x+qy)...(x+q^{r-1}y) \tag{71}$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{x^r y^r}{(r)!\,(c+r-1)_r (c)_{2r}} q^{rc+2r(r-1)}\, {}_0\Phi_1(c+2r;x)\, {}_0\Phi_1(c+2r;q^r y;\tfrac{1}{2}). \tag{72}$$

On replacing x, y, c, q by $-x^2$, $-y^2$, $\nu+1$, q^2 and, as before, using [ ] to denote factorials advancing by q^2, we obtain, from the series (71),

$$[2\nu]! \sum_{r=0}^{\infty} (-)^r \frac{(x^2+y^2)(x^2+q^2y^2)...(x^2+q^{2r-2}y^2)}{[2r]!\,[2\nu+2r]!}.$$

For x^2, y^2, q^2 substitute $R^2e^{-i\theta}/2$, $R^2e^{i\theta}/2$, $e^{i\omega}$. We then have

$$\frac{[2\nu]!}{R^\nu} \sum_{r=0}^{\infty} \frac{R^{\nu+2r}q^{r(r-1)/2}}{[2r]!\,[2\nu+2r]!} \cos\theta\cos(\theta+\omega)...\cos\{\theta+(r-1)\omega\}. \tag{73}$$

The expression (66) can be similarly transformed into Bessel's form, and we obtain finally

$$J_\nu^q\{R,\cos_r(\theta,\omega)\}$$
$$= \frac{R^\nu}{x^\nu y^\nu} \sum_{r=0}^{\infty} (-)^r \frac{[2\nu+2r-2]!\,[2\nu+4r]}{[2r]!} q^{2r\nu+4r^2-2r} J_{\nu+r}^2(x)\mathfrak{J}_{\nu+r}^q(y), \tag{74}$$

$$J_\nu^q(x) = \sum_{r=0}^{\infty} (-)^r \frac{x^{\nu+2r}}{[2r]!\,[2\nu+2r]!}, \quad \mathfrak{J}_\nu^2(y) = \sum_{r=0}^{\infty} (-)^r \frac{y^{\nu+2r}}{[2r]!\,[2\nu+2r]!} q^{3r^2-r}.$$

By similar transformations of (39) we obtain

$$J_\nu^q(x)\mathfrak{J}_\nu^q(y) = \sum_{r=0}^{\infty} (-)^r \frac{(xy/R)^{\nu+2r}}{[2\nu+2r]!\,[2r]!} q^{(r,\nu)} J_{\nu+2r}^\nu\{R,\cos_r(\theta+r\omega,\omega)\}, \tag{75}$$

where the index $(r,\nu) = 4r^2-3r/2$.

From lemma (12) we easily deduce

$$J_m^q(x)\mathfrak{J}_n^q(x) = \sum_{r=0}^{\infty} (-)^r \frac{[2m]!\,[2n]!\,\{2m+2n\}_r\{2m+2n+2\}_r}{[2r]!\,[2m+2]_r[2n+2]_r[2m+2n+2]_r} x. \tag{76}$$

Here { } denotes factorials advancing by q^4.

Theorems similar to (72), (73), (74), (75) but obtained by other methods have been given previously by the present writer.*

* Jackson, (4), (5).

7. Basic analogue of $F(a, b; c, x+y-xy)$

Burchnall and Chaundy have given several interesting expansions of this exceptional form of hypergeometric function. I give here some basic analogues. We need the following lemmas which I owe to Dr. Bailey: they are derived from a theorem due to G. N. Watson.*

$$ {}_3\Phi_2\left[\begin{matrix}(b-h),\ (-\theta),\ (-\phi)\\ (1-a-\theta-\phi),\ (b)\end{matrix};q\right] $$

$$ = \Delta_q(a)\,{}_3\Phi_2\left[\begin{matrix}(h),\ (-\theta),\ (-\phi)\\ (a),\ (b)\end{matrix},\ q^{a+b-h+\theta+\phi}\right], \tag{77} $$

from which, on replacing a, b, h by b, $1-a-\theta-\phi$, $c-a$, we obtain

$$ {}_3\Phi_2\left[\begin{matrix}(1-c-\theta-\phi),\ (-\theta),\ (-\phi)\\ (1-a-\theta-\phi),\ (1-b-\theta-\phi)\end{matrix},\ q\right] $$

$$ = \Delta_q(b)\,{}_3\Phi_2\left[\begin{matrix}(c-a,\ (-\theta),\ (-\phi)\\ (1-a-\theta-\phi),\ (b)\end{matrix},\ q^{b-c-1}\right]; \tag{78} $$

also, if in (77) we replace h by $a+b-c$, we obtain

$$ {}_3\Phi_2\left[\begin{matrix}(c-a),\ (-\theta),\ (-\phi)\\ (1-a-\theta-\phi),\ (b)\end{matrix},\ q\right] $$

$$ = \Delta_q(a)\,{}_3\Phi_2\left[\begin{matrix}(a+b-c),\ (-\theta),\ (-\phi)\\ (a),\ (b)\end{matrix},\ q^{c+\theta+\phi}\right]. \tag{79} $$

It is well to note that the functions on the right and left of (78), (79) respectively differ only in the solitary q-factors. From Watson's theorem we also derive a special case

$$ \Delta_q(c)\,{}_3\Phi_2\left[\begin{matrix}(a+b-c),\ (-\theta),\ (-\phi)\\ (a),\ (b)\end{matrix},\ q^{c+\theta+\phi}\right] $$

$$ = \sum_{r=0}^{\infty} \frac{(c-1)_r[c+1]_r(c-a)_r(c-b)_r(-\theta)_r(-\phi)_r}{(r)!\,(a)_r(b)_r[c-1]_r(c+\theta)_r(c+\phi)_r} q^{r(a+b+\theta+\phi)}. \tag{80} $$

Applying (75) to $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}(m)!\,(n)!} q^{n(n-1)/2}$

we obtain by the method given in detail in (29)

$$ \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r}{(r)!\,(c)_r} x^r y^r q^{r(a+b-c)+r(r-1)} \Phi(a+r, b+r; c+r, x|+q^r y, \tfrac{1}{2}) \tag{81} $$

$$ = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(c-a)_r}{(r)!\,(c)_{2r}} x^r y^r q^{r(a+b-c)+r(r-1)} \times $$

$$ \times \Phi(a+r, b+r, b+r; c+2r, x, q^r y, \tfrac{1}{2}). \tag{82} $$

* Watson (8).

When $|q| < 1$, the series (81) is absolutely convergent, so that on replacing the functions Φ by expansions and collecting terms diagonally we obtain

$$\sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(r)!\,(c)_r}\{(x|+q^r y)!-xyq^{a+b-c},\,1\}_r, \tag{83}$$

in which the variable of term $r+1$ is a polynomial analogous to the expansion of $(x+y-xy)^r$, namely

$$\sum_{s=0}^{r} \frac{(r)!}{(s)!\,(r-s)!}(x|+q^r y)_{r-s}\,x^s y^s q^{s(a+b-c)+s(s-1)}. \tag{84}$$

By applying lemmas (79), (80) we obtain the following equivalent series:

$$\sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(c-a)_r}{(r)!\,(c)_{2r}} x^r y^r q^{ra+r(r-1)}\Phi(a+r, b+r, b+r; c+2r; x, q^r y; \tfrac{1}{2}) \tag{85}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(a+b-c)_r}{(r)!\,(c)_{2r}} x^r y^r q^{rc+r(r-1)}\times$$

$$\times\Phi(a+r, a+r, b+r, b+r, c+2r; x, q^r y; \tfrac{1}{2}) \tag{86}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-1)_r[c+1]_r(c-a)_r(c-b)_r}{(r)!\,(c)_{2r}(c)_{2r}[c-1]_r} x^r y^r q^{r(a+b)+3r(r-1)/2}\times$$

$$\times\Phi(a+r, b+r; c+2r; x)\Phi(a+r, b+r, c+2r; q^r y; \tfrac{1}{2}). \tag{87}$$

The equations (81), (85), (87) reduce when $q = 1$ to the direct expansions of $F(a, b; c; x+y-xy)$ given by Burchnall and Chaundy. The forms of inverse results are easy to conjecture, but the only possible proofs appear to be by verification of terms, involving much tedious algebra. It does not appear worth while to give them.

8. Function of higher order

Consider

$$\nabla_q(h)\Delta_q(k)\Phi(a, b; c; x)\Phi(a', b'; c'; y, \lambda)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(h)_{m+n}(k)_m(a)_m(b)_m(k)_n(a')_n(b')_n}{(m)!\,(n)!\,(k)_{m+n}(h)_m(c)_m(h)_n(c')_n} x^m y^n q^{\lambda n(n-1)}$$

$$= \Phi\begin{bmatrix} h\colon k, a, b; k, a', b'; & x, y; \lambda \\ k\colon h, c; h, c'; & \end{bmatrix}$$

in the convenient notation used by Burchnall and Chaundy. Applying the operator series equivalent to $\nabla_q(h)\Delta_q(k)$, we obtain

$$\sum_{r=0}^{\infty}\sum_{m-r=0}^{\infty}\sum_{n-r=0}^{\infty}\frac{(k-h)_r(k)_{2r}(-m)_r(-n)_r(h)_{m+n}(k)_m(a)_m(b)_m(k)_n(a')_n(b')_n}{(r)!\,(k+r-1)_r(k+m)_r(k+n)_r(h)_r(k)_{m+n}(c)_m(h)_n(c')_n}\times$$
$$\times x^m y^n q^{r(k+m+n)+r(r-1)/2+\lambda n(n-1)+r(-m-n)+r(r-1)}.$$

Remembering that, when $\min(m,n) < r$, $(-m)_r(-n)_r$ vanishes, we put $M+r$, $N+r$ for m, n and obtain

$$\sum_{r=0}^{\infty}\sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\frac{(k-h)_r(k)_{2r}(h)_{M+N+2r}(k)_{M+r}}{(r)!\,(k+r-1)_r(k+M+r)_r(k+N+r)_r}\times$$
$$\times\frac{(a)_{M+r}(b)_{M+r}(k)_{N+r}(a')_{N+r}(b')_{N+r}}{(h)_r(k)_{M+N+2r}(c)_{M+r}(h)_{N+r}(c')_{N+r}}x^{M+r}y^{N+r}q^{rk+(\lambda+\frac{1}{2})r(r-1)+\lambda N(N-1)+2\lambda Nr},$$

whence by reduction of factorials such as $(a)_{M+r} = (a)_r(a+r)_M$ we find the final expression

$$\sum_{r=0}^{\infty}\frac{(k-h)_r(k)_r^2(a)_r(b)_r(a')_r(b')_r}{(r)!\,(k+r-1)_r(k)_{2r}(h)_r(c)_r(c')_r}x^r y^r q^{rk+(\lambda+\frac{1}{2})r(r-1)}\times$$
$$\times{}_3\Phi_2\left[\begin{matrix}k+r, a+r, b+r\\ k+2r, c+r\end{matrix};x\right]{}_3\Phi_2\left[\begin{matrix}k+r, a'+r, b'+r\\ k+2r, c'+r\end{matrix};yq^{2r\lambda};\lambda\right] \tag{88}$$

Other expansions analogous to those given by Burchnall and Chaundy can be found.

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CERTAIN EXPANSIONS OF SOLUTIONS OF THE HEUN EQUATION

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1. RECENTLY* I pointed out that solutions of the Fuchsian differential equation of the second order with four singularities can be expanded in certain series of hypergeometric functions. In the present paper I consider, more generally, all expansions of a certain type of solutions of the Heun equation in terms of hypergeometric functions. The result is that all such expansions must essentially, i.e. apart from transformations of the Heun equation or of the hypergeometric functions in question, be series either of the type considered earlier by Svartholm† or of the type dealt with in my former paper. Thus both types of series are now shown to arise from a systematic theory. The significant features of these two types of series are discussed, and it is shown that there is a connexion between series of two different types representing the same Heun function: the coefficients of the two series are essentially the same.

2. All branches of the function defined by the Riemannian scheme

$$P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & z \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{Bmatrix}, \tag{2.1}$$

where the exponents are connected by Riemann's relation

$$\alpha+\beta-\gamma-\delta-\epsilon+1 = 0, \tag{2.2}$$

satisfy the differential equation

$$\Lambda[y] \equiv z(z-1)(z-a)\left\{\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right)\frac{dy}{dz}\right\}+\alpha\beta(z-h)y = 0 \tag{2.3}$$

which will be called the Heun equation.‡ For certain exceptional values of the 'accessory parameter' h, two (in general distinct) branches of (2.1), e.g. the branch regular at $z = 0$ and that regular at $z = 1$, coincide: such an exceptional solution will be called a *Heun*

* A. Erdélyi, *Duke Math. J.* 9 (1942), 48–58.

† N. Svartholm, *Math. Ann.* 116 (1939), 413–21.

‡ K. Heun, *Math. Ann.* 33 (1889), 161–79.

function. If necessary, we specify the Heun function by stating, e.g., that it belongs to the exponent 0 both at $z = 0$ and at $z = 1$.

3. We shall try to find a solution of the Heun equation in the form of a series of hypergeometric functions. A hypergeometric function is a branch of a Riemann P-function with three singularities: these three singularities we shall choose at three of the four singularities of (2.1), for instance, at 0, 1, ∞. At two of the three singularities, e.g. at 0 and 1, the exponents of the Riemann P-function may be made to coincide with those of (2.1) so that we shall expect an expansion of the form

$$P = \sum_{m=0}^{\infty} c_m P_m, \tag{3.1}$$

where

$$P_m = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \lambda+m & z \\ 1-\gamma & 1-\delta & \mu-m & \end{Bmatrix}, \tag{3.2}$$

and where, since the sum of the exponents of a Riemann P-function of the second order with three singularities must be unity,

$$\lambda+\mu = \gamma+\delta-1 = \alpha+\beta-\epsilon. \tag{3.3}$$

Our object is to find out whether an expansion of the form (3.1) is possible, and, if it is, what are the admissible values of λ and μ.

4. Six significant branches of P_m are

$$P_m^1 = \frac{(-)^m\Gamma(\lambda+m)}{\Gamma(1-\mu+m)} F(\lambda+m, \mu-m; \gamma; z),$$

$$P_m^2 = \frac{(-)^m\Gamma(1-\gamma+\lambda+m)}{\Gamma(\gamma-\mu+m)} z^{1-\gamma} F(1-\gamma+\lambda+m, 1-\gamma+\mu-m; 2-\gamma; z),$$

$$P_m^3 = \frac{\Gamma(\lambda+m)\Gamma(1-\gamma+\lambda+m)}{\Gamma(1-\mu+m)\Gamma(\gamma-\mu+m)} F(\lambda+m, \mu-m; \delta; 1-z),$$

$$P_m^4 = (1-z)^{1-\delta} F(1-\delta+\lambda+m, 1-\delta+\mu-m; 2-\delta; 1-z),$$

$$P_m^5 = \frac{\Gamma(\lambda+m)\Gamma(1-\gamma+\lambda+m)}{\Gamma(1+\lambda-\mu+2m)} z^{-\lambda-m} \times$$
$$\times F\left(\lambda+m, 1-\gamma+\lambda+m; 1+\lambda-\mu+2m; \frac{1}{z}\right),$$

$$P_m^6 = \frac{\Gamma(\lambda-\mu+2m)}{\Gamma(1-\mu+m)\Gamma(\gamma-\mu+m)} z^{-\mu+m} \times$$
$$\times F\left(\mu-m, 1-\gamma+\mu-m; 1-\lambda+\mu-2m; \frac{1}{z}\right). \tag{4.1}$$

We shall assume throughout that

$$P_m = \sum_{\nu=1}^{6} \pi_\nu P_m^\nu, \tag{4.2}$$

where the constant coefficients π_ν do not depend on m (though they may depend on the exponents $\alpha,\ldots, \epsilon, \lambda, \mu$). A function of the form (4.2) satisfies the differential equation

$$z(z-1)\left\{\frac{d^2P_m}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}\right)\frac{dP_m}{dz}\right\}+(\lambda+m)(\mu-m)P_m = 0 \tag{4.3}$$

and the functional equation

$$\epsilon z(z-1)\frac{dP_m}{dz}+\{\alpha\beta(z-h)-(\lambda+m)(\mu-m)(z-a)\}P_m$$
$$= K_{m+1}P_{m+1}+L_m P_m+M_{m-1}P_{m-1}, \tag{4.4}$$

where

$$K_m = \frac{(m+\alpha-\mu-1)(m+\beta-\mu-1)(m+\gamma-\mu-1)(m-\mu)}{(2m+\lambda-\mu-1)(2m+\lambda-\mu-2)},$$

$$L_m = a(\lambda+m)(\mu-m)-h\alpha\beta+$$
$$+\frac{(m+\alpha-\mu)(m+\beta-\mu)(m+\gamma-\mu)(m+\lambda)}{(2m+\lambda-\mu)(2m+\lambda-\mu+1)}+$$
$$+\frac{(m-\alpha+\lambda)(m-\beta+\lambda)(m-\gamma+\lambda)(m-\mu)}{(2m+\lambda-\mu)(3m+\lambda-\mu-1)},$$

$$M_m = \frac{(m-\alpha+\lambda+1)(m-\beta+\lambda+1)(m-\gamma+\lambda+1)(m+\lambda)}{(2m+\lambda-\mu+1)(2m+\lambda-\mu+2)}. \tag{4.5}$$

In fact, (4.3) is the differential equation satisfied by all branches of (3.2). We can prove (4.4) for P_m^1 and P_m^2 by substitution of the power series and comparison of powers of z: the extension of (4.4) to any P_m of the form (4.2) follows from the remark that any such P_m is a linear combination, with coefficients independent of m, of P_m^1 and P_m^2.

In order to discuss the convergence of our series, some knowledge of the asymptotic properties of P_m for large m is needed. From Watson's results* it is easily found that

$$\lim_{m\to\infty}\frac{P_{m+1}^5}{P_m^5} = \frac{1-(1-1/z)^{\frac{1}{2}}}{1+(1-1/z)^{\frac{1}{2}}}, \tag{4.6}$$

* G. N. Watson, *Proc. Cambridge Phil. Soc.* 22 (1918), 277–308.

while, whenever P_m is not a multiple of P_m^5,

$$\lim_{m\to\infty} \frac{P_{m+1}}{P_m} = \frac{1+(1-1/z)^{\frac{1}{2}}}{1-(1-1/z)^{\frac{1}{2}}} \tag{4.7}$$

in both cases provided that $\Re(1-1/z)^{\frac{1}{2}} > 0$.

5. Let us now try to expand a solution of the Heun equation in a series

$$y = \sum_{m=0}^{\infty} c_m P_m, \tag{5.1}$$

where the c_m are certain constant coefficients. From (2.3),

$$\Lambda[y] = z(z-1)(z-a)\left\{\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}\right)\frac{dy}{dz}\right\}+(z-a)(\lambda+m)(\mu-m)y+$$
$$+\epsilon z(z-1)\frac{dy}{dz}+\{\alpha\beta(z-h)-(\lambda+m)(\mu-m)(z-a)\}y,$$

and hence, from (4.3) and (4.4),

$$\Lambda\left[\sum_{m=0}^{\infty} c_m P_m\right] = \sum_{m=0}^{\infty} c_m\{K_{m+1}P_{m+1}+L_m P_m+M_{m-1}P_{m-1}\}.$$

If $c_0 \neq 0$, the differential equation $\Lambda[\sum c_m P_m] = 0$ thus implies the conditions

$$M_{-1}P_{-1} = 0, \tag{5.2}$$

$$L_0 c_0+M_0 c_1 = 0, \tag{5.3}$$

$$K_m c_{m-1}+L_m c_m+M_m c_{m+1} = 0 \quad (m = 1, 2, 3,\ldots). \tag{5.4}$$

Here (5.2), together with (3.3), determines the admissible values of λ and μ and in each case the admissible branches of the P_m: (5.3) and (5.4) are a system of recurrence relations for the coefficients c_m.

6. We defer the discussion of (5.2) to the next section, and study in the present section the convergence of our series.

If $m \to \infty$, we have

$$\lim 4K_m/m^2 = \lim 4M_m/m^2 = 1 \quad \text{and} \quad \lim 4L_m/m^2 = 2-4a,$$

and hence we infer from a well-known theorem of Poincaré's on linear difference equations* that $t = \lim c_{m+1}/c_m$ exists and satisfies the quadratic equation $t^2+(2-4a)t+1 = 0$, provided that the two roots t_1 and t_2 of this quadratic equation have different moduli, i.e. provided that a is not a real number between 0 and 1. Determine

* H. Poincaré, *American J. of Math.* 7 (1885), 203–58. Cf. also Milne-Thomson, *Calculus of Finite Differences*, p. 527.

$(1-1/a)^{\frac{1}{2}}$ uniquely by the convention $\Re(1-1/a)^{\frac{1}{2}} > 0$. Then the two roots are

$$t_1 = \frac{1+(1-1/a)^{\frac{1}{2}}}{1-(1-1/a)^{\frac{1}{2}}}, \qquad t_2 = 1/t_1, \tag{6.1}$$

t_1 being the root with the larger modulus.

In general, i.e. if h is not a root of the transcendental equation (6.3) *infra*, Poincaré's theorem furthermore shows that $\lim c_{m+1}/c_m = t_1$ and from (6.1), (4.6), and (4.7) it is seen that in this case $\sum c_m P_m$ diverges unless P_m is a multiple of P_m^5: $\sum c_m P_m^5$ converges for

$$\left|\frac{1-(1-1/z)^{\frac{1}{2}}}{1+(1-1/z)^{\frac{1}{2}}}\right| < \left|\frac{1-(1-1/a)^{\frac{1}{2}}}{1+(1-1/a)^{\frac{1}{2}}}\right|, \tag{6.2}$$

i.e. if z is outside the ellipse which has foci at $z = 0$ and $z = 1$ and which passes through the third finite singularity, $z = a$. Outside this ellipse $\sum c_m P_m^5$ is convergent and represents a solution of the Heun equation that belongs to the exponent λ at infinity.

In the exceptional case, when h is a root of the transcendental equation

$$L_0/M_0 - \frac{K_1/M_1}{L_1/M_1 -} \frac{K_2/M_2}{L_2/M_2 - \dots} = 0, \tag{6.3}$$

Poincaré's theorem asserts that $\lim c_{m+1}/c_m = t_2$ and it is then seen from (6.1), (4.6), and (4.7) that in this exceptional case $\sum c_m P_m$ is convergent for

$$\left|\frac{1+(1-1/z)^{\frac{1}{2}}}{1-(1-1/z)^{\frac{1}{2}}}\right| < \left|\frac{1+(1-1/a)^{\frac{1}{2}}}{1-(1-1/a)^{\frac{1}{2}}}\right|, \tag{6.4}$$

i.e. inside the said ellipse, with the possible exception of the line joining the foci of this ellipse: $\sum c_m P_m$ represents, in this domain, the general solution of the Heun equation. In this case $\sum c_m P_m^5$ is convergent in the whole z-plane cut along the line joining $z = 0$ to $z = 1$, with the possible exception of the cut itself. In this cut z-plane $\sum c_m P_m^5$ represents the Heun function which in this case arises from the coincidence of the branch regular at $z = a$ and the branch belonging to the exponent λ at infinity.

7. We now come to the discussion of (5.2). If

$$\lambda = \alpha, \qquad \mu = \beta - \epsilon \tag{7.1}$$

or if

$$\lambda = \beta, \qquad \mu = \alpha - \epsilon, \tag{7.2}$$

then $M_{-1}P_{-1}$ vanishes, for all branches of P_m, identically in z. The series arising from any of the above two admissible values of λ and μ will be called series of type (I). Take, for instance, (7.1). In the

general case, i.e. if h is not a root of the transcendental equation (6.3) with $\lambda = \alpha$, $\mu = \beta-\epsilon$, the branch belonging to the exponent α at $z = \infty$ will be represented by a series of type (I), namely $\sum c_m P_m^5$, convergent in the domain (6.2). Since a suitable linear transformation brings any singularity to infinity, our result is that every solution of the Heun equation can be represented by a series of type (I). In fact, the series introduced in my former paper are easily seen to be of type (I).

In the exceptional case, i.e. if h satisfies (6.3) with $\lambda = \alpha$, $\mu = \beta-\epsilon$, the above-mentioned branch becomes a Heun function, regular at $z = a$ and belonging to the exponent α at infinity: this Heun function is represented by a series of type (I) convergent in the plane cut along the line joining the two singularities $z = 0$, $z = 1$ of the Heun function. As in the general case, a suitable linear transformation shows that every Heun function may be represented by a series of type (I) convergent in the whole plane cut along a straight line or a circular arc joining the two singularities of the Heun function. The general solution of the Heun equation in this case may be represented by a series of type (I) convergent in the domain (6.4).

8. If

$$\lambda = \gamma+\delta-1, \qquad \mu = 0, \tag{8.1}$$

$$\lambda = \gamma, \qquad \mu = \delta-1; \tag{8.2}$$

$$\lambda = \delta, \qquad \mu = \gamma-1, \tag{8.3}$$

or

$$\lambda = 1, \qquad \mu = \gamma+\delta-2, \tag{8.4}$$

then three branches of P_m, of which one is always P_m^6, coincide and for these three branches $M_{-1}P_{-1}$ vanishes. The other three branches, of which one is always P_m^5, also coincide with one another, but for these latter three branches $M_{-1}P_{-1}$ does not vanish. The series arising from these values of λ, μ will be called series of type (II). It follows that series of type (II) are necessarily of the form $\sum c_m P_m^6$. Hence there can be no series of type (II) in the general case, for P_m^6 is not a multiple of P_m^5.

Let us consider the exceptional case in which h satisfies (6.3) with the values (8.1) of λ and μ, say. In this case P_m^6 is a polynomial of degree m in z and the series $\sum c_m P_m^6$ is convergent in the domain (6.4) there representing the Heun function which is regular at $z = 0$ and at $z = 1$. Similarly, in the other three cases (8.2), (8.3), and (8.4) there is only one series of type (II) in each case, representing

a Heun function belonging respectively to the pairs of exponents $0,\ 1-\delta;\ 1-\gamma,\ 0$; and $1-\gamma,\ 1-\delta$ at $z=0$ and $z=1$. It follows by a linear transformation that every Heun function, but no other solution of the Heun equation, can be represented by a series of type (II). In each of the four cases (8.1)–(8.4), P_m^6 is either a Jacobi polynomial or can be expressed in terms of Jacobi polynomials. In fact, the series introduced by Svartholm are of type (II).

9. Comparing now the two types of series, we see that the first type is more general in its application. Even in the case of Heun functions, when both types may be used, the first type is superior to the second in that its domain of convergence is more extensive than that of the series of the second type, and in that, unlike type (II), it is capable of representing the general solution of the Heun equation.

The chief advantage of type (II) is that series of this type are *orthogonal* series and this property is useful in some applications, e.g. when Heun functions must be normalized.

Let us consider the Heun function regular at $z=0$ and $z=1$. Its expansion of type (II) is of the form

$$y=\sum A_m F(-m,\gamma+\delta+m-1;\gamma;z). \tag{9.1}$$

The Riemannian scheme of the Heun equation may alternatively be represented as

$$(a-z)^{1-\epsilon}P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha-\epsilon+1 & z \\ 1-\gamma & 1-\delta & \epsilon-1 & \beta-\epsilon+1 & \end{Bmatrix}, \tag{9.2}$$

and hence our Heun function has the alternative expansion

$$y=(a-z)^{1-\epsilon}\sum B_m F(-m,\gamma+\delta+m-1;\gamma;z). \tag{9.3}$$

Now, it is known that

$$\int\limits_0^1 z^{\gamma-1}(1-z)^{\delta-1}F(-m,\gamma+\delta+m-1;\gamma;z)F(-m',\gamma+\delta+m'-1;\gamma;z)\,dz$$

$$=0 \quad \text{if } m\neq m', \quad \text{and}$$

$$=\frac{(-)^m m!\,\Gamma(\delta+m)\{\Gamma(\gamma)\}^2}{(\gamma+\delta+2m-1)\Gamma(\gamma+\delta+m-1)\Gamma(\gamma+m)} \quad \text{if } m=m'.$$

Hence for the integral which is to be normalized we obtain

$$\int\limits_0^1 z^{\gamma-1}(1-z)^{\delta-1}(a-z)^{\epsilon-1}y^2\,dz$$

$$=\sum_{m=0}^{\infty}\frac{(-)^m\{\Gamma(\gamma)\}^2\Gamma(\delta+m)m!\,A_m B_m}{(\gamma+\delta+2m-1)\Gamma(\gamma+\delta+m-1)\Gamma(\gamma+m)}. \tag{9.4}$$

The recurrence relations determine A_1, A_2,... in terms of A_0 and B_1, B_2,... in terms of B_0. The relation $\sum A_m = \sum B_m a^{1-\epsilon}$ in conjunction with (9.4) determines the actual numerical values for the normalized Heun function. The transcendental equations for h connected with the expansions (9.1) and (9.3) are the same so that the ratios of the A_m and the ratios of the B_m are found from the same continued fraction.

10. Finally, it is worth while pointing out that there is a relation between a series of type (I) and one of type (II) representing the same Heun function: the coefficients in the one are multiples of the coefficients of the other. Let us again briefly consider the Heun function regular at $z = 0$ and $z = 1$. It will be sufficient to consider the case where $\Re\gamma > 0$, $\Re\delta > 0$; in the more general case the real integrals must be replaced by contour integrals, but the essence of the procedure is the same.

If $\Re\gamma > 0$, $\Re\delta > 0$, the Heun function $F(z)$, regular at $z = 0$ and $z = 1$, satisfies the integral equation*

$$F(z) = \lambda\left(1-\frac{z}{a}\right)^{\epsilon-1} \int\limits_0^1 \zeta^{\gamma-1}(1-\zeta)^{\delta-1} F\left(\alpha-\epsilon+1, \beta-\epsilon+1; \gamma; \frac{z\zeta}{a}\right) F(\zeta)\, d\zeta. \tag{10.1}$$

Using the expansion (9.1) on the right-hand side of this integral equation, we have

$$F(z) = \lambda\left(1-\frac{z}{a}\right)^{\epsilon-1} \sum A_m \int\limits_0^1 \zeta^{\gamma-1}(1-\zeta)^{\delta-1} \times$$
$$\times F\left(\alpha-\epsilon+1, \beta-\epsilon+1; \gamma; \frac{z\zeta}{a}\right) F(-m, \gamma+\delta+m-1; \gamma; \zeta)\, d\zeta, \tag{10.2}$$

and, if the integration is performed, the expansion

$$F(z)$$
$$= \lambda'\left(1-\frac{z}{a}\right)^{\epsilon-1} \sum (-)^m \frac{\Gamma(\alpha-\epsilon+m+1)\Gamma(\beta-\epsilon+m+1)\Gamma(\delta+m)}{\Gamma(\gamma+m)\Gamma(\gamma+\delta+2m)} A_m \times$$
$$\times \left(\frac{z}{a}\right)^m F\left(\alpha-\epsilon+m+1, \beta-\epsilon+m+1; \gamma+\delta+2m; \frac{z}{a}\right) \tag{10.3}$$

emerges, which is in fact an expansion of type (I) connected with the Riemannian scheme (9.2), which is, as we have seen, equivalent to (2.1). The coefficients of (10.3) are multiples of the coefficients of (9.1).

* A. Erdélyi, *Quart. J. of Math.* (Oxford), 13 (1942), 107–12.

FIRST AND SECOND VARIATIONS OF THE LENGTH INTEGRAL IN A GENERALIZED METRIC SPACE

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1. Preliminary

THE metric space considered is n-dimensional, with coordinates x^i (h, i, j, k run from 1 to n throughout), and will be denoted by S_n; to each point of the space is associated a contravariant vector-density of weight p having components u^i, called the *element of support*. In the special case $p = 0$ the S_n becomes a Finsler space.

$L(x, u)$ is a scalar which is positive and homogeneous of first degree in the u, called the *fundamental function* of the geometry. The equations

$$a_{ij} = (\tfrac{1}{2}L^2)_{;ij} \tag{1.1}$$

(where $;i$ indicates differentiation with respect to u^i) define a tensor density of weight $-2p$, whilst the determinant

$$|a_{ij}| = a \tag{1.2}$$

is a density of weight $-2(np-1)$; then the g_{ij} defined by

$$g_{ij} = a_{ij}\, a^{-p/(np-1)} \tag{1.3}$$

are the components of a tensor, and we take them for metric coefficients.†

The vector $\vec{l}$ is the vector of unit length in the direction of the element of support. Its components are given by‡

$$l^i = \frac{u^i}{L\sqrt{g^p}}. \tag{1.4}$$

If $f(x, u)$ is a function homogeneous in the u, we define $f\|_i$ by

$$f\|_i = L\sqrt{(g^p)}f_{;i}, \tag{1.5}$$

which is also homogeneous in the u and of the same degree as f itself. Defining the tensor

$$A_{ijk} = \tfrac{1}{2}g_{ij}\|_k \tag{1.6}$$

and the vector

$$A_i = \tfrac{1}{2}(\log g)\|_i \tag{1.7}$$

† Following Schouten and Haantjes, (**4**), 163.

‡ For proof see (**2**).

and indicating contraction with respect to l by a 0, we see that†

$$A_{ij0} = 0, \tag{1.8}$$

$$A_0 = 0, \tag{1.9}$$

$$A_{0ij} = pl_i A_j. \tag{1.10}$$

The absolute differential of a tensor is defined by an expression similar to that used in Finsler geometry: e.g., for a vector $\overrightarrow{X}$,

$$DX^i = dX^i + X^j \omega_j^{\ i}, \tag{1.11}$$

$$DX_i = dX_i - X_j \omega_i^{\ j},$$

where‡

$$\omega_i^{\ j} = dx^k \Gamma_i{}^j{}_k + du^k C_i{}^j{}_k \tag{1.12}$$

and

$$C_{ijk} = \tfrac{1}{2} g_{ij;k}. \tag{1.13}$$

Writing ϖ^k for Dl^k, as an alternative to (1.12), we have§

$$\omega_i^{\ j} = dx^k \overset{*}{\Gamma}_i{}^j{}_k + \varpi^k A_i{}^j{}_k. \tag{1.14}$$

A vector $\overrightarrow{X}$ is said to be transported by *parallelism* when it is displaced so that $DX^i = 0$.

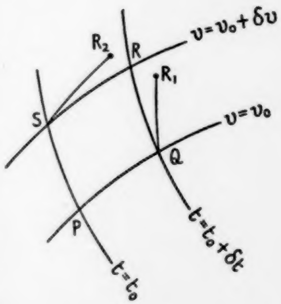
2. Torsion and curvature of the space

Consider a 2-space in the S_n defined by equations of form

$$x^i = x^i(t, v),$$

at each point of which the element of support is defined by equations of form

$$u^i = u^i(t, v).$$



$PQRS$ is an infinitesimal figure bounded by the curves $t = t_0$, $t = t_0 + \delta t$, $v = v_0$, $v = v_0 + \delta v$.

When the infinitesimal vectors $\overrightarrow{PS}$, $\overrightarrow{PQ}$ are transported by parallelism from P to Q, and from P to S respectively, vectors $\overrightarrow{QR_1}$, $\overrightarrow{SR_2}$ are obtained, and, in general, the points R_1 and R_2 will not coincide. The vector $\overrightarrow{R_1 R_2}$ we call the *torsion* of the space at P corresponding to the mesh $PQRS$, and it

† For proof see (**2**). ‡ For the values of the $\Gamma_i{}^j{}_k$ see (**4**).

§ For the values of the $\overset{*}{\Gamma}_i{}^j{}_k$ see (**4**). They are symmetrical in the lower indices.

will have components $\Omega^i\,\delta t\delta v$ to the second order in δt, δv where†

$$\Omega^i = \left[\frac{dx^j}{dt}\,\frac{\omega_j{}^i}{dv}\right], \tag{2.1}$$

where d/dt, D/dt, d/dv, D/dv are partial operators (ordinary and absolute), and the brackets [ ] indicate that from the expression contained therein we subtract the corresponding expression obtained by interchange of t, v.

From (2.1) it follows that

$$\Omega^i = \left[\frac{D}{dv}\,\frac{dx^i}{dt}\right] \quad \text{since} \quad \left[\frac{d^2x^i}{dvdt}\right] = 0. \tag{2.2}$$

Further, if a vector $\overrightarrow{X}$ is transported by parallelism round the figure $PQRSP$, the difference between the initial and final values of the components is given by $\Omega_j{}^i X^j \delta t\delta v$ to the second order in δt, δv where†

$$\Omega_j{}^i = \left[\frac{\omega_j{}^k}{dt}\,\frac{\omega_k{}^i}{dv} - \frac{d}{dt}\,\frac{\omega_j{}^i}{dv}\right]. \tag{2.3}$$

We call $\Omega_j{}^i\,\delta t\delta v$ the *curvature* of the space at P corresponding to the mesh $PQRS$.

If $\overrightarrow{T}$ is a vector defined at each point of the 2-space by equations of form

$$T^i = T^i(t, v),$$

then

$$\frac{D}{dv}\,\frac{DT^i}{dt} = \frac{d}{dv}\left(\frac{dT^i}{dt} + T^j\frac{\omega_j{}^i}{dt}\right) + \left(\frac{dT^k}{dt} + T^j\frac{\omega_j{}^k}{dt}\right)\frac{\omega_k{}^i}{dv}.$$

From this and (2.3) we obtain

$$\left[\frac{D}{dv}\,\frac{D}{dt}\right]T^i = \Omega_j{}^i T^j \quad \text{since} \quad \left[\frac{d^2T^j}{dvdt}\right] = 0. \tag{2.4}$$

In (2.1), writing

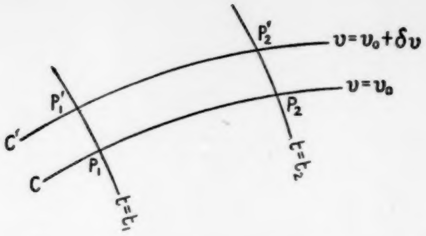
$$\frac{\omega_j{}^i}{dv} = \frac{dx^k}{dv}\overset{*}{\Gamma}{}_j{}^i{}_k + \frac{\varpi^k}{dv}A_j{}^i{}_k,$$

we obtain

$$\Omega^i = \left[\frac{dx^j}{dt}\,\frac{\varpi^k}{dv}\right]A_j{}^i{}_k. \tag{2.5}$$

† The proof of this is identical with that for a Finsler space: see (**1**), § 34.

3. Formulae for the first and second variations of the length integral



C is a curve joining two points P_1, P_2 in S_n. We consider any 2-space through C defined by equations of form

$$x^i = x^i(t, v),$$

at each point of which the element of support is defined by equations of form

$$u^i = u^i(t, v)$$

and the coordinates t, v in the 2-space are chosen so that C is the curve $v = v_0$, the parameter t along C having values t_1, t_2 at P_1, P_2.

C' is a neighbouring curve given by $v = v_0 + \delta v$ and P_1', P_2' are the points on C' for which $t = t_1$, t_2. The length of C between P_1, P_2 is thus given by

$$I = \int_{t_1}^{t_2} F\,dt, \quad \text{where} \quad F^2 = g_{ij}(x, u)\frac{dx^i}{dt}\frac{dx^j}{dt}.$$

The length of C' between P_1', P_2' is similarly given by

$$I' = \int_{t_1}^{t_2} \left(F + \delta v\frac{dF}{dv} + \tfrac{1}{2}\delta v^2\frac{d^2F}{dv^2} + \ldots\right) dt,$$

so that $I' - I = \delta I + \delta^2 I + \ldots$, where

$$\delta I = \delta v \int \frac{dF}{dv}\,dt, \quad \text{the \textit{first variation},}$$

$$\delta^2 I = \tfrac{1}{2}\delta v^2 \int_{t_1}^{t_2} \frac{d^2F}{dv^2}\,dt, \quad \text{the \textit{second variation}, etc.}$$

For abbreviation I shall put

$$\frac{dx^i}{dt} = \lambda^i, \qquad \frac{dx^i}{dv} = \mu^i,$$

and ξ^j for the unit vector in the former direction. Lengths of vectors μ^i, $D\lambda^i/dv$, etc., I denote by μ, λ_v, etc. I call μ^i the *displacement vector*.

To evaluate δI we have

$$\frac{dF}{dv} = \frac{1}{2F}\frac{dF^2}{dv} = \frac{1}{2F}\frac{DF^2}{dv} = \frac{1}{F}g_{ij}\frac{D\lambda^i}{dv}\lambda^j,$$

whilst $\lambda^i = F\xi^i$, so that

$$\frac{dF}{dv} = \frac{D\lambda^i}{dv}\xi^i. \tag{3.1}$$

In the present notation (2.2) and (2.5) may be written

$$\frac{D\lambda^i}{dv} - \frac{D\mu^i}{dt} = \Omega^i, \tag{3.2}$$

$$\Omega^i = \left(\lambda^j\frac{\varpi^k}{dv} - \mu^j\frac{\varpi^k}{dt}\right)A_{j\ k}^{\ i}. \tag{3.3}$$

From (3.1) and (3.2) we get

$$\frac{dF}{dv} = \left(\frac{D\mu^i}{dt} + \Omega^i\right)\xi_i, \tag{3.4}$$

and, putting $t =$ arc length s along C, this gives

$$\boxed{\delta I = \delta v \int\limits_{s_1}^{s_2} \left(\frac{D\mu^i}{ds} + \Omega^i\right)\xi_i\, ds.} \tag{3.5}$$

Now $$\frac{d}{dt}(\mu^i\xi_i) = \frac{D}{dt}(\mu^i\xi_i) = \frac{D\mu^i}{dt}\xi_i + \mu^i\frac{D\xi_i}{dt}.$$

From (3.5) we now obtain

$$\boxed{\delta I = \delta v[\mu^i\xi_i]_{s_1}^{s_2} - \delta v \int\limits_{s_1}^{s_2} \left(\mu^i\frac{D\xi_i}{ds} - \Omega^i\xi_i\right) ds.} \tag{3.6}$$

To evaluate $\delta^2 I$ we have from (3.1), (3.4)

$$\frac{d^2F}{dv^2} = \frac{D}{dv}\frac{dF}{dv} = \frac{D}{dv}\left\{\left(\frac{D\mu^i}{dt} + \Omega^i\right)\xi_i\right\}$$

$$= \left(\frac{D^2\mu^i}{dvdt} + \frac{D\Omega^i}{dv}\right)\xi_i + \frac{\frac{D\lambda^i}{dv}\frac{D\lambda_i}{dv}}{F} - \frac{\left(\frac{D\lambda^i}{dv}\xi_i\right)^2}{F}.$$

The last two terms together give

$$\frac{\lambda_v^2}{F}\sin^2(\lambda_v, \xi)$$

$\left(\text{unless the } \dfrac{D\lambda^i}{dv} = 0, \text{ when they vanish}\right)$, (λ_v, ξ) denoting the angle between vectors $D\lambda^i/dv$, ξ^i. Hence

$$\boxed{\delta^2 I = \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \left\{\left(\frac{D^2\mu^i}{dvds} + \frac{D\Omega^i}{dv}\right)\xi_i + \lambda_v^2 \sin^2(\lambda_v, \xi)\right\} ds.} \tag{3.7}$$

Now

$$\frac{d}{dt}\left(\frac{D\mu^i}{dv}\xi_i\right) = \frac{D}{dt}\left(\frac{D\mu^i}{dv}\xi_i\right) = \frac{D^2\mu^i}{dtdv}\xi_i + \frac{D\mu^i}{dv}\frac{D\xi_i}{dt}$$
$$= \left(\frac{D^2\mu^i}{dvdt} - \mu^j\Omega_j{}^i\right)\xi_i + \frac{D\mu^i}{dv}\frac{D\xi_i}{dt},$$

since $\dfrac{D^2\mu^i}{dvdt} - \dfrac{D^2\mu^i}{dtdv} = \mu^j\Omega_j{}^i$ from (2.4). From (3.7) we now obtain

$$\boxed{\begin{aligned}\delta^2 I = \tfrac{1}{2}\delta v^2 \left[\frac{D\mu^i}{dv}\xi_i\right]_{s_1}^{s_2} - \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \left(\frac{D\mu^i}{dv}\frac{D\xi_i}{ds} - \frac{D\Omega^i}{dv}\xi_i\right) ds + \\ + \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \{\lambda_v^2 \sin^2(\lambda_v, \xi) - \xi^i\mu^j\Omega_{ij}\}\, ds.\end{aligned}} \tag{3.8}$$

4. Conditions for an extremal

The curve C of § 3 will be said to be an *extremal* between P_1, P_2 if $\delta I = 0$, and $\mu^i = 0$ at P_1, P_2. The displacement of the curve and the element of support between the end-points may be arbitrary or subject to certain restrictions.

From (3.3) and (3.6), when $\mu^i = 0$ at P_1, P_2,

$$\delta I = -\delta v \int\limits_{s_1}^{s_2} \left\{\mu^i\left(\frac{D\xi_i}{ds} + \xi^j \frac{\varpi^k}{ds} A_{ijk}\right) - \frac{\varpi^k}{dv}\xi^i\xi^j A_{ijk}\right\} ds. \tag{4.1}$$

The conditions that $\delta I = 0$ for values of μ^i, ϖ^k/dv arbitrary save for satisfying $\dfrac{\varpi^k}{dv} l_k$ are

$$\frac{D\xi_i}{ds} + \xi^j \frac{\varpi^k}{ds} A_{ijk} = 0, \qquad \xi^i\xi^j A_{ijk} = 0 \quad \text{since} \quad A_{ij0} = 0.$$

Hence

$$\boxed{\begin{array}{l}\textit{C is extremal for arbitrary displacement of curve and}\\ \textit{element of support if and only if}\\ \left.\begin{array}{ll}\text{(i)} & \dfrac{D\xi^i}{ds}+\xi^j\dfrac{\varpi^k}{ds}A^i{}_{jk}=0\\ \text{(ii)} & \xi^j\xi^k A_{jki}=0\end{array}\right\}\ \textit{along } C.\end{array}} \tag{4.2}$$

That these equations do define a curve through a given point O of S_n in a direction satisfying (4.2) (ii), along which the vector $\vec{l}$ is determined, may be seen as follows:

The $\overset{*}{\Gamma}{}_j{}^i{}_k$ and the $A_j{}^i{}_k$ are homogeneous of zero order in the u^i and will therefore be unaltered if we replace the u^i by the l^i. Then (4.2) (i) are n equations linear in d^2x^i/ds^2 and dl^i/ds, whilst the remaining terms involve the x^i, dx^i/ds, and l^i. In (4.2) (ii) there are only $n-1$ independent equations, since $l^iA_{jki}=0$; differentiation with respect to s of any $n-1$ of these equations gives $n-1$ further equations of a similar type. We now have $2n-1$ equations in the $2n-1$ unknowns d^2x^i/ds^2, dl^i/ds (the latter comprising only $n-1$ independent unknowns since they are related through $\varpi^i l_i=0$). Thus the d^2x^i/ds^2 and dl^i/ds may be expressed in terms of the x^i, dx^i/ds, and l^i; further differentiation enables d^3x^i/ds^3, d^4x^i/ds^4,... and d^2l^i/ds^2, d^3l^i/ds^3,... to be similarly expressed. The coordinates of the point on the extremal through O in a given direction $\xi_0^i=(dx^i/ds)_0$ satisfying $\xi_0^j\,\xi_0^k(A_{jki})_0=0$, arc distance s from 0, and the components at this point of the unit vector $\vec{l}$, are then determined by

$$x^i=(x^i)_0+s\left(\frac{dx^i}{ds}\right)_0+\frac{s^2}{2}\left(\frac{d^2x^i}{ds^2}\right)_0+\dots,$$

$$l^i=(l^i)_0+s\left(\frac{dl^i}{ds}\right)_0+\frac{s^2}{2}\left(\frac{d^2l^i}{ds^2}\right)_0+\dots.$$

In general, (4.2) (ii) is not satisfied when the element of support is tangential to C, but, if, as a special case, we suppose the element of support to be restricted in this way, then (4.2) (ii) becomes

$$l^jl^kA_{jki}=0.$$

Since $A_{00i} = pA_i$, we must now have $A_i = 0$ if $p \neq 0$. Then (4.2) (i) becomes

$$\frac{\varpi^i}{ds}+pl^i\frac{\varpi^k}{ds}A_k = 0$$

since $A^i{}_{0k} = pl^iA_k$, and finally $\dfrac{\varpi^i}{ds} = 0$ since $A_k = 0$. Hence

> *If the element of support is tangential to C along C, C is extremal for arbitrary displacement of curve and element of support if and only if* (4.3)
> (i) *C is autoparallel,*
> (ii) *A_i vanishes along C.*

If we next suppose that the element of support is tangential, not only to C but to all the curves v = constant, the displacement of the element of support is restricted, and we now have the following analysis.

From (3.2) and (3.3)

$$\frac{D\lambda^i}{dv}-\frac{D\mu^i}{dt} = \Omega^i = \left(\lambda^j\frac{\varpi^k}{dv}-\mu^j\frac{\varpi^k}{dt}\right)A_j{}^i{}_k;$$

hence, if $\dfrac{1}{F}\lambda^i = \xi^i = l^i$ for variation of v,

$$\frac{\varpi^i}{dv} = \frac{1}{F}\frac{D\lambda^i}{dv}+\lambda^i\frac{d}{dv}\frac{1}{F}$$

and
$$\frac{\varpi^i}{dv}-\frac{D\mu^i}{dt} = \frac{D\lambda^k}{dv}A_0{}^i{}_k-\mu^j\frac{\varpi^k}{dt}A_j{}^i{}_k,$$

since $A_j{}^i{}_0 = 0$; on rearrangement this gives

$$\frac{D\lambda^k}{dv}(\delta^i_k-pl^iA_k) = \frac{D\mu^i}{dt}-\mu^j\frac{\varpi^k}{dt}A_j{}^i{}_k$$

and, making use of

$$(\delta^i_k-pl^iA_k)(\delta^h_i+pl^hA_i) = \delta^h_k,$$

we may solve for $D\lambda^h/dv$, obtaining

$$\frac{D\lambda^h}{dv} = \frac{D\mu^i}{dt}(\delta^h_i+pl^hA_i)-\mu^j\frac{\varpi^k}{dt}A_j{}^i{}_k(\delta^h_i+pl^hA_i),$$

whence, from (3.2),

$$\Omega^h = p\frac{D\mu^i}{dt}l^hA_i-\mu^j\frac{\varpi^k}{dt}A_j{}^i{}_k(\delta^h_i+pl^hA_i)$$

and
$$\Omega^hl_h = p\frac{D\mu^i}{dt}A_i-\mu^i\frac{\varpi_k}{dt}(A_{i0}{}^k+pA_i{}^{jk}A_j).$$

Using this and $$\frac{d}{dt}(\mu^i A_i) = \frac{D\mu^i}{dt}A_i + \mu^i\frac{DA_i}{dt},$$

we now get from (3.6), that

$$\delta I = -\delta v \int_{s_1}^{s_2} \mu^i \left\{(\delta_i^k + pl_i A^k + pA_i^{\ jk}A_j)\frac{\varpi^k}{ds} + p\frac{DA_i}{ds}\right\} ds \tag{4.4}$$

when $\mu^i = 0$ at P_1, P_2.

Thus $\delta I = 0$ for arbitrary μ^i if $\dfrac{\varpi_k}{ds} = 0$ and $\dfrac{DA_i}{ds} = 0$. Hence†

If the element of support is tangential to the curves $v =$ constant, C is extremal for variations consistent with this restriction if

(i) *C is autoparallel,*

(ii) *A_i is transported by parallelism along C.*

These conditions are sufficient, but not necessary. (4.5)

We next consider the corresponding results for the particular case of a Finsler space. Both conditions of (4.2) are now satisfied if $\xi^i = \pm l^i$ and $\dfrac{D\xi^i}{ds} = 0$. We shall next inquire whether it is possible for C to be extremal in a Finsler space for arbitrary displacement of the element of support, when the latter is *not* tangential to C along C—a possibility which certainly exists in a contravariant vector-density space, as can be seen from (4.2).

The equations

$$A_{jki}\xi^j\xi^k = 0, \qquad g_{jk}\xi^j\xi^k = 1 \tag{4.6}$$

in the ξ^i are satisfied by $\xi^i = \pm l^i$, and we shall inquire whether any other solution is possible. The A_{ijk} are related by the equations

$$l^i A_{ijk} = 0 \tag{4.7}$$

and an A_{ijk} is unaltered by an interchange of suffixes. We transform to another set of coordinates x'^i, so chosen that, at a given point, the components of the unit vector l'^i are given by

$$l'^1 \neq 0, \qquad l'^i = 0 \quad \text{if} \quad i \neq 1.$$

† This result has already been obtained by Davies by a different method. See (**2**), § 10.

Equations (4.6) and (4.7) then transform into

$$A'_{jki}\xi'^j\xi'^k = 0, \qquad g'_{jk}\xi'^j\xi'^k = 1, \tag{4.8}$$

$$l'^i A'_{ijk} = 0. \tag{4.9}$$

If the $\xi'^2,\ldots,\xi'^n$ all vanish, (4.8) gives $g'_{11}(\xi'^1)^2 = 1$, and since $g'_{11}(l'^1)^2 = 1$, we have $\xi'^1 = \pm l'^1$, whence $\xi'^i = \pm l'^i$ and $\xi^i = \pm l^i$; if, then, equations (4.6) have a solution other than $\xi^j = \pm l^j$, the $\xi'^2,\ldots,\xi'^n$ do not all vanish. Then (4.9) reduces to $A'_{1jk} = 0$, i.e. every A'_{ijk} involving suffix 1 vanishes, and (4.8) now gives

$$A'_{jki}\xi'^j\xi'^k = 0 \quad (i,j,k = 2,3,\ldots,n).$$

This is a set of $n-1$ quadratic equations in the $n-2$ ratios $\xi'^2:\xi'^3:\ldots:\xi'^n$ from which the $\xi'^2,\ldots,\xi'^n$ can be eliminated if they do not all vanish; we then obtain a relation between the A'_{ijk} not involving the suffix 1.

Thus, in order that equations (4.6) may have a solution other than $\xi^i = \pm l^i$, it is necessary that the A_{ijk} be restricted. It may be noted that, in the case $n = 2$, the relation between the A'_{ijk} reduces to $A'_{222} = 0$; since A'_{111}, A'_{112}, A'_{122} all vanish because they involve suffix 1, the space is now Riemannian.

If the element of support is tangential to C, the first condition of (4.2) reduces to $\dfrac{D\xi^i}{ds} = 0$, whence

> *In a Finsler space, C is extremal for arbitrary displacement of curve and element of support if*
>
> (i) *C is autoparallel,*
>
> (ii) *the element of support is tangential to C along C.* (4.10)
>
> *These conditions are sufficient; they are also necessary unless the equations* $A_{jki}\xi^j\xi^k = 0$ *have a solution other than* $\xi^i = \pm l^i$, *and then the* A_{jki} *are subject to certain restrictions.*

If, as a special case, we suppose the element of support in the Finsler space tangential to C along C, (4.1) reduces to

$$\delta I = -\delta v \int_{s_1}^{s_2} \mu^i \frac{D\xi_i}{ds}\, ds$$

and $\delta I = 0$ for arbitrary values of μ^i if and only if $\dfrac{D\xi_i}{ds} = 0$. This

is true if the displacement of the element of support is arbitrary or subject to certain restrictions. Hence

In a Finsler space, if the element of support is tangential to C along C, C is extremal for displacement of the element of support either (α) *arbitrary, or* (β) *restricted by the latter being tangential to the curves* $v =$ *constant, if and only if C is autoparallel.*	(4.11)

Thus, comparing the above results, if C is a curve along which the element of support is tangential, the conditions that it be an extremal

(α) for arbitrary displacement of element of support,

(β) for displacement of the element of support restricted by the latter being tangential to the curves $v =$ constant,

are the same for a Finsler space, from (4.11), but different for a contravariant vector-density space, from (4.3) and (4.5).

5. Theorems on the first and second variations of the length integral

In order to facilitate comparison, corresponding theorems for Riemannian† and contravariant vector-density spaces will be written side by side; we shall find that, in certain cases, the Riemannian theorems are true also for a Finsler space.

Since a Riemannian space is characterized by $A_{ijk} = 0$, in this case $\Omega^i = 0$ and (3.5) and (3.6) reduce to

$$\delta I = \delta v \int\limits_{s_1}^{s_2} \frac{D\mu^i}{ds} \xi_i \, ds, \tag{5.1}$$

$$\delta I = \delta v [\mu^i \xi_i]_{s_1}^{s_2} - \delta v \int\limits_{s_1}^{s_2} \mu^i \frac{D\xi_i}{ds} \, ds. \tag{5.2}$$

For the first variation we then have the following comparison.

THEOREM I

Riemannian space	*Contravariant vector-density space*
The first variation vanishes if $\frac{D\mu^i}{ds}\xi_i = 0$ *along C* [*from* (5.1)].	*The first variation vanishes if* $\left(\frac{D\mu^i}{ds} + \Omega^i\right)\xi_i = 0$ *along C* [*from* (3.5)].

Next suppose C is an extremal. The type of variation of the

† The results for a Riemannian space have already been given by Synge: see (**5**).

element of support must be specified, e.g. the variation may be arbitrary as in (4.2), or restricted by a certain condition as in (4.5). In either case the integral in (3.6), which is the same as that in (4.1), vanishes for arbitrary μ^i, and so we have

THEOREM II

The first variation vanishes if C is extremal and the displacement vector is normal to C at the end points [*from* (5.2)].

If C is extremal, the first variation vanishes for the same type of variation of the element of support as that for which C is extremal, if the displacement vector is normal to C at the end points [*from* (3.6)].

THEOREM III

The first variation vanishes if the displacement vector is normal to the principal normal of C along C, and is normal to C or vanishes at the end points [*from* (5.2)].

The first variation vanishes if the angle between the displacement vector and the principal normal of C is determined by $\mu^i\dfrac{D\xi_i}{ds} = \Omega^i\xi_i$ *along C, and the displacement vector is normal to C or vanishes at the end points* [*from* (3.6)].

In the Riemannian case, (3.7) and (3.8) reduce to

$$\delta^2 I = \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \left\{\frac{D^2\mu^i}{dvds}\xi_i + \mu_s^2 \sin^2(\mu_s, \xi)\right\} ds, \tag{5.3}$$

$$\delta^2 I = \tfrac{1}{2}\delta v^2 \left[\frac{D\mu^i}{dv}\xi_i\right]_{s_1}^{s_2} - \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \frac{D\mu^i}{dv}\frac{D\xi_i}{ds}\, ds + {}$$
$$+ \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \{\mu_s^2 \sin^2(\mu_s, \xi) - \xi^i\mu^j\xi^h\mu^k R_{ijhk}\}\, ds. \tag{5.4}$$

For the second variation we then have the theorems

THEOREM IV

The second variation is positive, zero, negative according as

$$\frac{D^2\mu^i}{dvds}\xi_i + \mu_s^2 \sin^2(\mu_s, \xi)$$

is positive, zero, negative along C [*from* (5.3)].

The second variation is positive, zero, negative according as

$$\left(\frac{D^2\mu^i}{dvds} + \frac{D\Omega^i}{dv}\right)\xi_i + \lambda_v^2 \sin^2(\lambda_v, \xi)$$

is positive, zero, negative along C [*from* (3.7)].

THEOREM V

C being extremal, the second variation is positive for variation with fixed ends if

$$\xi^i\mu^j\xi^k\mu^h R_{ijkh} \leqslant 0$$

along C [from (5.4)].

C being extremal for conditions as in (4.3), the second variation is positive for variation with fixed ends if

$$\xi^i\mu^j\Omega_{ij} \leqslant 0 \quad \textit{and} \quad \frac{D\Omega^i}{dv}\xi_i \geqslant 0$$

along C [from (3.8)].

We lastly consider certain special cases for a Finsler space.

The above theorems given for a Riemannian space hold also for a Finsler space under the following conditions:

(α) *I, II, III, if the element of support is tangential to C along C;*

(β) *I, II, III, V, if the element of support is tangential to curves v = constant;*† (5.5)

(γ) *all the theorems, if the element of support is tangential to curves v = constant and $\dfrac{D\mu^i}{ds}\Omega_i = 0$ along C;*

(δ) *all the theorems, if the curves t = constant are extremals with tangential element of support.*

The proof is as follows:

In (α), $\Omega^i\xi_i = 0$ since $A_j{}^i{}_k l_i = 0$ in a Finsler space; (3.5) and (3.6) then reduce to (5.1) and (5.2).

In (β), which includes (α), $\Omega^i\lambda_i = 0$ for variation of v, whence $\dfrac{D\Omega^i}{dv}\lambda_i+\Omega_i\dfrac{D\lambda^i}{dv} = 0$; (3.7) and (3.8) now become

$$\delta^2 I = \tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \left\{\frac{D^2\mu^i}{dvds}\xi_i+\mu_s^2\sin^2(\mu_s,\xi)+\frac{D\mu^i}{ds}\Omega_i\right\} ds, \tag{5.6}$$

$$\delta^2 I = \tfrac{1}{2}\delta v^2\left[\frac{D\mu^i}{dv}\xi_i\right]_{s_1}^{s_2} - $$

$$-\tfrac{1}{2}\delta v^2 \int\limits_{s_1}^{s_2} \left\{\frac{D\mu^i}{dv}\frac{D\xi_i}{ds}+\xi^i\mu^j\Omega_{ij}+\mu_s^2\sin^2(\mu_s,\xi)+\frac{D\mu^i}{ds}\Omega_i\right\} ds. \tag{5.7}$$

† This case has been fully treated by Householder: see (**3**).

If C is extremal as in the Riemannian theorem V, then along C $\frac{\varpi^k}{ds} = 0$, $\lambda^j A_j{}^i{}_k = 0$, whence

$$\Omega^i = \left(\lambda^j \frac{\varpi^k}{dv} - \mu^j \frac{\varpi^k}{ds}\right) A_j{}^i{}_k = 0,$$

and (5.7) reduces to (5.4) with $\frac{D\xi_i}{ds} = 0$, and so the Riemannian theorem V now holds.

In (γ), which includes (β), since $\frac{D\mu^i}{ds}\Omega_i = 0$ along C, (5.6) reduces to (5.3), whence the Riemannian theorem IV now holds. It may be noted that $\frac{D\mu^i}{ds}\Omega_i = 0$ along C if the displacement vector is transported by parallelism along C, or if the vector $D\mu^i/ds$ is tangential to C (and therefore in the direction of the element of support).

In (δ), by an argument similar to that used in (β), $\Omega^i = 0$ along every curve $t =$ constant, and therefore everywhere in the 2-space. Equations (3.5), (3.6), (3.7), (3.8) now reduce to the corresponding Riemannian formulae (5.1), (5.2), (5.3), (5.4). Further, (5.2) now gives $\frac{D\xi_i}{ds} = 0$ as the condition that C be extremal for variation of the element of support restricted by the latter's being tangential to the curves $t =$ constant, and all the Riemannian theorems now hold.

In conclusion, I wish to thank Dr. E. T. Davies, to whom I am greatly indebted for his most helpful suggestions.

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THE FLAT REGIONS OF INTEGRAL FUNCTIONS OF FINITE ORDER

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1. THE term 'flat region' in connexion with regular and, in particular, integral functions, is frequently used to denote a region in which the minimum modulus is in some sense of the same order as the maximum modulus. The following theorem was proved by J. M. Whittaker.

THEOREM A.† *There is an absolute constant H, not less than* 16/729, *with the following property. If $f(x)$ is an integral function of order $\rho \leqslant 2$, satisfying the condition $n(r) = o(r^2)$, and if h $(< H)$, η $(< \rho)$, and d are given positive numbers, then the values of ζ for which the inequality*

$$\log|f(z)| > h \log M(|\zeta|) > |\zeta|^{\eta}, \qquad |z-\zeta| \leqslant d,$$

is satisfied form a set of upper density greater than or equal to $(H-h)/(1-h)$.

A theorem for integral functions of any finite order was also obtained by J. M. Whittaker.

THEOREM B.‡ *Let $f(z)$ be an integral function of order ρ, and let σ be a given number less than $(1-\frac{1}{2}\rho)$. Then there is a positive constant h and a sequence ζ_1, ζ_2, ζ_3,..., such that $|\zeta_s| \to \infty$ and*

$$\log|f(z)| > h \log M(|\zeta_s|),$$

in the circle $|z-\zeta_s| \leqslant \lambda_s^{\sigma}$.

The above results have been extended to meromorphic functions (and also slightly improved for integral functions) by A. J. MacIntyre, who proves the following theorem.

THEOREM C.§ *Hypothesis: $f(z)$ is a meromorphic function of finite positive order ρ; ρ' $(\geqslant \rho)$ is any number such that*

$$T(r) \equiv T(r,f) = o\left(\frac{r^{\rho'}}{\log r}\right),$$

and the poles are of deficiency $\delta > 0$, so that

$$\overline{\lim_{r\to\infty}} \frac{N(r,\infty)}{T(r)} \leqslant 1-\delta.$$

† J. M. Whittaker (**6**). ‡ J. M. Whittaker (**5**). § A. J. MacIntyre (**1**).

Assertion: we can find a sequence of circles C_ν with centres ξ_ν and radii r_ν satisfying the conditions

$$\lim_{\nu\to\infty} |\xi_\nu| = \infty, \qquad \lim_{\nu\to\infty} r_\nu |\xi_\nu|^{\frac{1}{2}\rho'-1} = \infty,$$

and a corresponding sequence of positive numbers ϵ_ν satisfying $\lim \epsilon_\nu = 0$, *such that for all z in C_ν, $f(z)$ satisfies the inequalities*

$$1-\epsilon_\nu < \frac{\log|f(z)|}{\log|f(\xi_\nu)|} < 1+\epsilon_\nu,$$

and
$$\frac{\delta}{e} < \frac{\log|f(\xi_\nu)|}{T(|\xi_\nu|, f)} < |\xi_\nu|^{\rho'}.$$

Since for an integral function $\delta = 1$ and

$$T(|\xi_\nu|, f) > \frac{k-1}{k+1} \log M\left(\frac{|\xi_\nu|}{k}\right),$$

for any $k > 1$, Theorem C gives for integral functions

$$\log|f(z)| > (1-\epsilon_\nu)\frac{k-1}{k+1}\frac{1}{e}\log M\left(\frac{|\xi_\nu|}{k}\right),$$

for z in C_ν.

In a later paper A. J. MacIntyre and R. Wilson consider in detail the value of $|f'(z)/f(z)|$, and $|D^q \log f(z)|$, where $f(z)$ is a meromorphic function. The majority of their results are not primarily concerned with regions in which $|f(z)|$ is large, but the following theorem is an exception to this statement.

THEOREM D.† *If $f(z)$ is a meromorphic function of finite order, whose poles are of permanent defect, then there exist arbitrarily large values z_1 such that*

$$|\log f(z) - \log f(z_1)| \leqslant \frac{A\,T(r)}{r}|z-z_1|$$
$$\leqslant AC\sqrt{\{T(r)\}},$$

and
$$\log|f(z_1)| > B\,T(r),$$

where $|z_1| = r$ and z is any point on the circle $|z-z_1| < Cr/\sqrt{T(r)}$, and A, B, C are positive constants depending only on the order of $f(z)$ and the defect of the poles.

For an integral function of order ρ,

$$\overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \rho;$$

† A. J. MacIntyre and R. Wilson (2).

so that the above theorem gives that, if $f(z)$ is an integral function of order ρ, and $\epsilon > 0$, there exist arbitrarily large z_1 such that

$$|\log f(z) - \log f(z_1)| \leqslant ACr^{\frac{1}{2}(\rho+\epsilon)},$$

and, for any $k > 1$,

$$\log|f(z_1)| > B\frac{k-1}{k+1}\log M\left(\frac{r}{k}\right),$$

provided that z lies on the circle $|z-z_1| = r_\nu$, where

$$r_\nu r^{\frac{1}{2}\rho-1} < C\sqrt{\left\{\frac{(k+1)r^{\frac{1}{2}\rho}}{(k-1)\sqrt{\{\log M(r/k)\}}}\right\}}.$$

In the case of integral functions the defect of the poles is 1, and the constants A, B, C depend only on the order of the function. They have no relation to the type of the function.

The purpose of this paper is to prove the following results.

THEOREM I. *Suppose that the integral function $f(z)$ is of order two and mean type κ. Suppose also that d is less than $\frac{1}{2}\sqrt{(\pi/2\kappa)}$.*

Then there exists a sequence of points $\zeta_1, \zeta_2, \ldots$, such that $|\zeta_s| \to \infty$, and that, given any $\epsilon > 0$,

$$\log|f(z)| > (\kappa-\epsilon)|z|^2,$$

provided that $|z-\zeta_s| < d$, and $s > s(\epsilon)$.

By a similar method of proof a result is obtained on the flat regions of functions regular in an angle, and this is embodied in Theorem II.

THEOREM II. *Suppose that $f(z)$ is regular and of order $\rho > 0$ in the angle $|\arg z| \leqslant \alpha < \pi/2\rho$, and that*

$$\varlimsup_{r\to\infty}\frac{\log M\{f(z), r, \alpha\}}{r^\rho} = \kappa > 0,$$

where $$M\{f(z), r, \alpha\} = \max|f(re^{i\theta})| \quad (|\theta| \leqslant \alpha).$$

Suppose also that $\varlimsup_{r\to\infty}\dfrac{\log|f(re^{\pm i\alpha})|}{r^\rho} < \kappa$, *and that* $d < \frac{1}{2}$.

Then there exists a sequence of points $\zeta_1, \zeta_2, \zeta_3, \ldots$ within the angle and distant at least $\dfrac{2d}{\rho}\sqrt{\left(\dfrac{\pi}{2\kappa}\right)}|\zeta_s|^{1-\frac{1}{2}\rho}$ from its arms such that

$$|\zeta_1| \leqslant |\zeta_2| \leqslant |\zeta_3| \leqslant \ldots \to \infty,$$

and that, if $\epsilon > 0$, $$\log|f(z)| > (\kappa-\epsilon)|z|^\rho,$$

provided that $$|z-\zeta_s| < \left(\frac{2d}{\rho}\right)\sqrt{\left(\frac{\pi}{2\kappa}\right)}|\zeta_s|^{1-\frac{1}{2}\rho},$$
for some $s > s'(\epsilon)$.

The result for functions regular in an angle leads directly to the following theorem for integral functions of any finite order.

THEOREM III. *Suppose that $f(z)$ is an integral function of order $\rho > 0$ and type $\kappa > 0$. Suppose also that $h(\theta_0) = \kappa$, where*
$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{r^\rho}.$$
Then, if $\epsilon > 0$, $\delta > 0$, $d < \frac{1}{2}$, there is within the angle
$$|\arg z - \theta_0| < \delta$$
a sequence of circles
$$|z-\zeta_s| < \frac{2d}{\rho}\sqrt{\left(\frac{\pi}{2\kappa}\right)}|\zeta_s|^{1-\frac{1}{2}\rho}, \quad \textit{where } |\zeta_s| \to \infty,$$
such that $\log|f(z)| > (\kappa-\epsilon)|z|^\rho$, *provided that z is within a circle for which $s > s''(\epsilon)$.*

The above theorem clearly includes Theorem I as a special case. As the proof of Theorem III involves the use of the Phragmén-Lindelöf theorem for functions regular in an angle, it seems worth while to indicate how, for functions of order two, the result may be obtained directly.

2. *Notation.* The square in the z-plane with vertices
$$\{(m-\tfrac{1}{2}\pm\tfrac{1}{2})+i(n-\tfrac{1}{2}\pm\tfrac{1}{2})\},$$
is called the square $S_{m,n}$, and $z_{m,n}$ is a point lying within or on the boundary of $S_{m,n}$. The integral function $F(z)$ is defined by the equation
$$F(z) \equiv e^{-\alpha z^2} z \prod_{m=-\infty}^{\infty} {\prod_{n=-\infty}^{\infty}}' \left\{\left(1-\frac{z}{z_{m,n}}\right)\exp\left(\frac{z}{z_{m,n}}+\frac{z^2}{2z_{m,n}^2}\right)\right\},$$
where $$2\alpha = \lim_{R\to\infty} \sum_{|m+in|\leqslant R}' \frac{1}{z_{m,n}^2},$$
and the dash denotes the omission of $(0,0)$ from the sum or product. Throughout this paper the double sequence $\{z_{m,n}\}$ will be subject to the further condition that any two members are distant at least $\rho_1 > 0$ apart.

3. I have proved elsewhere† that $F(z)$ is of order two and type $\frac{1}{2}\pi$, and that it also has the properties given in the following lemmas.

LEMMA 1. *Suppose that ϵ is any small positive number. Then*

$$\left|\frac{\log|F'(z_{m,n})|}{|z_{m,n}|^2}-\tfrac{1}{2}\pi\right| < \epsilon,$$

provided that $|z_{m,n}| > R(\epsilon)$, where $R(\epsilon)$ does not depend on the particular set $\{z_{m,n}\}$.

LEMMA 2. *Suppose that $f(z)$ is an integral function of order two and type less than $\frac{1}{2}\pi$. Then*

$$f(z) \equiv F(z)\left\{\sum{}' \frac{f(z_{m,n})}{(z-z_{m,n})F'(z_{m,n})}+\frac{f(0)}{z}\right\}.$$

LEMMA 3. *Suppose that the double sequence of numbers $\{Z_{m,n}\}$ is such that*

$$\overline{\lim_{|z_{m,n}|\to\infty}} \frac{\log|Z_{m,n}|}{|z_{m,n}|^2} = \kappa' < \tfrac{1}{2}\pi.$$

Suppose also that the integral function $H(z)$ is defined by the equation

$$H(z) \equiv F(z)\sum_{m,n}{}' \left\{\frac{Z_{m,n}}{(z-z_{m,n})F'(z_{m,n})}\right\}.$$

Then $H(z)$ is of order two and type less than or equal to $\frac{1}{2}\pi$.

This last lemma has only been proved for $\{Z_{m,n}\}$ bounded, but the alterations in the proof are trivial.

LEMMA 4. *Suppose that $f(z)$ is an integral function of order 2 and type κ less than $\frac{1}{2}\pi$. Then*

$$\overline{\lim}\frac{\log|f(z_{m,n})|}{|z_{m,n}|^2} = \kappa.$$

Proof. Let $\kappa_1 = \overline{\lim}\dfrac{\log|f(z_{m,n})|}{|z_{m,n}|^2}$, so that $\kappa_1 \leqslant \kappa < \frac{1}{2}\pi$. Then, if η is a positive number less than $(\frac{1}{2}\pi-\kappa_1)$, there is $\lambda < 1$ such that

$$\kappa_1+\tfrac{1}{2}\pi\lambda^2+\eta = \tfrac{1}{2}\pi.$$

The function $F(\lambda z)$ is of order two and type $\frac{1}{2}\lambda^2\pi$. Suppose that

$$F(\lambda z) = a_0+a_1 z+a_2 z^2+a_3 z^3+..., \tag{3.1}$$

so that
$$\sum_{p=0}^{\infty}|a_p|\,|z_{m,n}|^p \leqslant \exp\{(\tfrac{1}{2}\lambda^2\pi+\epsilon)|z_{m,n}|^2\}, \tag{3.2}$$

† B. J. Maitland (3).

for some ϵ $(0 < \epsilon < \frac{1}{2}\eta)$ and large $|z_{m,n}|$. The triple series

$$\sum_{m,n,p}{}' \left\{\frac{f(z_{m,n})a_p z_{m,n}^p}{F'(z_{m,n})(z-z_{m,n})}\right\} \tag{3.3}$$

is seen to converge absolutely except for z equal to $z_{m,n}$, if we use the result given in Lemma 1. From Lemma 2 it follows, on replacing $f(z)$ by $\{f(z)z^h\}$, that

$$\frac{f(z)z^h}{F(z)} \equiv \sum_{m,n}{}' \left\{\frac{f(z_{m,n})z_{m,n}^h}{F'(z_{m,n})(z-z_{m,n})}\right\}. \tag{3.4}$$

Summing (3.3) in two different ways, and using (3.1), (3.4), we have

$$\frac{f(z)F(\lambda z)}{F(z)} \equiv \sum_{m,n}{}' \left\{\frac{f(z_{m,n})F(\lambda z_{m,n})}{F'(z_{m,n})(z-z_{m,n})}\right\},$$

that is
$$f(z)F(\lambda z) \equiv F(z) \sum_{m,n}{}' \left\{\frac{f(z_{m,n})F(\lambda z_{m,n})}{F'(z_{m,n})(z-z_{m,n})}\right\}. \tag{3.5}$$

The left-hand side of (3.5) is an integral function of order two and type $(\kappa+\frac{1}{2}\lambda^2\pi)$; since, if $h > 0$,

$$\left|\frac{\log|F(z)|}{|z|^2} - \tfrac{1}{2}\pi\right| < \epsilon', \tag{3.6}$$

provided that† $|z| > R'(\epsilon')$, and $|z-z_{m,n}| > h$. The magnitude of $|f(z)|$ must be greater somewhere on the boundary of $|z-z_{m,n}| = h$ than at any point within this circle, so that the set of excepted circles for which (3.6) does not hold cannot lower the order of $f(z)F(\lambda z)$ below the stated value. On the other hand, the result given in Lemma 3 shows that the order of the right-hand side of (3.5) is less than or equal to $\frac{1}{2}\pi$. Hence

$$\kappa+\tfrac{1}{2}\lambda^2\pi \leqslant \tfrac{1}{2}\pi,$$

that is
$$\kappa+(\tfrac{1}{2}\pi-\kappa_1-\eta) \leqslant \tfrac{1}{2}\pi,$$

i.e.
$$\kappa \leqslant \kappa_1+\eta.$$

Since η is any positive number, $\kappa \leqslant \kappa_1$. But this means that $\kappa = \kappa_1$, which is the required result.

4. Sufficient material has now been collected to prove Theorem I in the special case $\kappa = \frac{1}{2}\pi$.

LEMMA 5. *Suppose that $f(z)$ is an integral function of order two and type κ less than or equal to $\frac{1}{2}\pi$. Suppose also that d is any positive*

† B. J. Maitland (3).

number less than $\frac{1}{2}$. Then there exists a sequence of points $\zeta_1^*, \zeta_2^*, \zeta_3^*,...$ such that $|\zeta_s^*| \to \infty$, and that

$$\log|f(z)| > (\kappa-\epsilon)|z|^2, \tag{4.1}$$

provided that $|z-\zeta_s^*| < d$, for some ζ_s^* where $|\zeta_s^*| > R_0(\epsilon)$.

Proof. Let $S_{m,n}^*$ be a square of side $2d$, concentric with $S_{m,n}$, and having its edges parallel to those of $S_{m,n}$. Let $z_{m,n}$ be the point within or on the boundary of $S_{m,n}^*$ for which $|f(z)|$ is least.

From Lemma 4, if $\kappa < \frac{1}{2}\pi$,

$$\overline{\lim_{|z_{m,n}|\to\infty}} \frac{\log|f(z_{m,n})|}{|z_{m,n}|^2} = \kappa.$$

It is thus possible to select a sequence of squares, $\{S_{m_s,n_s}\}$ $(s = 1, 2, 3,...)$, where $|m_s+in_s|$ increases and tends to infinity with s, such that

$$\lim_{s\to\infty} \frac{\log|f(z_{m_s,n_s})|}{|z_{m_s,n_s}|^2} = \kappa.$$

Then, if $\epsilon > 0$, and $s > s_1(\epsilon)$,

$$\log|f(z_{m_s,n_s})| > (\kappa-\epsilon)|z_{m_s,n_s}|^2.$$

But, if z lies in S_{m_s,n_s}^*, $|f(z)| \geqslant |f(z_{m_s,n_s})|$, so that

$$\log|f(z)| > (\kappa-2\epsilon)|z|^2$$

provided that $s > s_2(\epsilon)$.

If ζ_s^* is the mid-point of S_{m_s,n_s}, then the circle $|z-\zeta_s^*| < d$ lies within S_{m_s,n_s}^*, and the lemma has been proved for $\kappa < \frac{1}{2}\pi$.

If $\kappa = \frac{1}{2}\pi$, let λ be such that $2d < \lambda < 1$, and let $f^*(z) \equiv f(\lambda z)$. Then $f^*(z)$ is of order two and type $\frac{1}{2}\lambda^2\pi < \frac{1}{2}\pi$. If $\epsilon > 0$, it follows from the above that there exists a sequence $\zeta_1, \zeta_2, \zeta_3,...$ such that

$$\log|f^*(z)| > (\tfrac{1}{2}\pi\lambda^2-\epsilon\lambda^2)|z|^2,$$

provided that $|z-\zeta_s| < d/\lambda < \frac{1}{2}$, and $s > s_2(\epsilon\lambda^2)$. That is

$$\log|f(z)| > (\tfrac{1}{2}\pi-\epsilon)|z|^2,$$

provided that $|z-\zeta_s^*| < d$, where $\zeta_s^* = \zeta\lambda$, and $|\zeta_s^*|$ is sufficiently large. This completes the proof of the lemma.

Proof of Theorem I. Let $f_1(z) \equiv f\{\sqrt{(\pi/2\kappa)}z\}$, where κ is the type of $f(z)$. Then $f_1(z)$ is an integral function of order two and type $\frac{1}{2}\pi$. From Lemma 5 there exists a sequence $\{\zeta_s^*\}$ $(s = 1, 2,...)$ such that $|\zeta_s^*| \to \infty$ and

$$\log|f_1(z)| > \left(\tfrac{1}{2}\pi-\frac{1}{2}\frac{\pi}{\kappa}\epsilon\right)|z|^2,$$

provided that $|z-\zeta_s^*| < \sqrt{(2\kappa/\pi)}d < \frac{1}{2}$, and $|\zeta_s| > R_0(\pi\epsilon/2\kappa)$, where d is any number $< \frac{1}{2}\sqrt{(\pi/2\kappa)}$.

Let $\zeta_s = \sqrt{(\pi/2\kappa)}\zeta_s^*$. Then the sequence $\{\zeta_s\}$ is such that $|\zeta_s| \to \infty$, and

$$|f(z)| \equiv \left|f_1\left\{\sqrt{\left(\frac{2\kappa}{\pi}\right)}z\right\}\right| > (\kappa-\epsilon)|z|^2,$$

provided that
$$\left|\sqrt{\left(\frac{2\kappa}{\pi}\right)}z-\zeta_s^*\right| < \sqrt{\left(\frac{2\kappa}{\pi}\right)}d,$$

or
$$|z-\zeta_s| < d,$$

and $|\zeta_s|$ sufficiently large. This completes the proof of the theorem.

5. A more general result emerges on considering functions regular in an angle. The following result has been obtained previously.

LEMMA 6.† *Suppose that the function $f(z)$ is regular in the angle $\delta \leqslant \arg z \leqslant \frac{1}{2}\pi-\delta$. Suppose also that*

$$\overline{\lim}\frac{\log\log M(r)}{\log r} = 2, \tag{i}$$

where $M(r) = \max|f(re^{i\theta})|$ $(\delta \leqslant \theta \leqslant \frac{1}{2}\pi-\delta)$,

$$\overline{\lim_{r\to\infty}}\frac{1}{r^2}\log|f(re^{i\delta})| < \tfrac{1}{2}\pi, \tag{ii}$$

and
$$\overline{\lim_{r\to\infty}}\frac{1}{r^2}\log|f(re^{i(\frac{1}{2}\pi-\delta)})| < \tfrac{1}{2}\pi,$$

$$\overline{\lim_{|z_{m,n}|\to\infty}}\frac{1}{|z_{m,n}|^2}\log|f(z_{m,n})| < \tfrac{1}{2}\pi, \tag{iii}$$

where there is one $z_{m,n}$ corresponding to every $S_{m,n}$ whose area within the angle is greater than $\frac{1}{2}$, and the distances of the $z_{m,n}$ from the arms of the angle and from each other exceed some positive ρ_1.

Then, for all θ such that $\delta \leqslant \theta \leqslant \frac{1}{2}\pi-\delta$,

$$\overline{\lim_{r\to\infty}}\frac{\log|f(re^{i\theta})|}{r^2} \leqslant \tfrac{1}{2}\pi,$$

the result holding uniformly in θ.

From the above result the following lemma is deduced.

† B. J. Maitland (3).

LEMMA 7. *Suppose that $f(z)$ is regular and of order two in the angle $|\arg z| \leqslant \alpha$, where $\alpha < \frac{1}{4}\pi$. Suppose also that*

$$\overline{\lim_{r\to\infty}} \frac{1}{r^2}\log|f(re^{\pm i\alpha})| < \tfrac{1}{2}\pi,$$

and that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,f)}{r^2} = \tfrac{1}{2}\pi,$$

where

$$M(r,f) = \max_{-\alpha\leqslant\theta\leqslant\alpha} |f(re^{i\theta})|.$$

Let d be any number less than $\frac{1}{2}$.

Then there exists a sequence of points $\zeta_1, \zeta_2, \zeta_3,\ldots$, distant at least d from the arms of the angle, such that

$$|\zeta_1| \leqslant |\zeta_2| \leqslant |\zeta_3| \leqslant \ldots,$$

and $|\zeta_s| \to \infty$ as $s \to \infty$, and that, given any $\epsilon > 0$,

$$\log|f(z)| > (\tfrac{1}{2}\pi-\epsilon)|z|^2,$$

provided that $|z-\zeta_s| < d$, where $s > s_0(\epsilon)$.

Proof. Let $g(z) \equiv f(\lambda e^{-\frac{1}{4}i\pi}z)$; so that $g(z)$ is regular in the angle $\delta \leqslant \arg z \leqslant \frac{1}{2}\pi-\delta$, where $\delta = (\frac{1}{4}\pi-\alpha)$. Also

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r,g)}{\log r} = 2, \qquad M(r,g) = \max_{\delta\leqslant\theta\leqslant\frac{1}{2}\pi-\delta} |g(re^{i\theta})|,$$

and

$$\overline{\lim_{r\to\infty}} \frac{\log|g(re^{i\delta})|}{r^2} = \lambda^2 \overline{\lim_{r\to\infty}} \frac{1}{r^2}\log|f(re^{-i\alpha})|;$$

$$\overline{\lim_{r\to\infty}} \frac{\log|g(re^{i(\frac{1}{2}\pi-\delta)})|}{r^2} = \lambda^2 \overline{\lim_{r\to\infty}} \frac{1}{r^2}\log|f(re^{i\alpha})|.$$

Let λ be chosen greater than 1 but sufficiently near 1 for the following to be true:

$$\overline{\lim_{r\to\infty}} \frac{1}{r^2}\log|g(re^{i\delta})| < \tfrac{1}{2}\pi, \tag{5.1}$$

and

$$\overline{\lim_{r\to\infty}} \frac{1}{r^2}\log|g(re^{i(\frac{1}{2}\pi-\delta)})| < \tfrac{1}{2}\pi. \tag{5.2}$$

Also

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,g)}{r^2} = \tfrac{1}{2}\pi\lambda^2 > \tfrac{1}{2}\pi. \tag{5.3}$$

Let $S^*_{m,n}$ be the square concentric with $S_{m,n}$, and of side $2d < 1$, the sides of $S_{m,n}$ and $S^*_{m,n}$ being parallel. Also let $z^*_{m,n}$ be the point in or on the boundary of $S^*_{m,n}$ (or that portion of $S^*_{m,n}$ within the angle $\delta \leqslant \arg z \leqslant \frac{1}{2}\pi-\delta$) for which $|g(z)|$ is least. A point $z_{m,n}$ is

thus defined for every square $S_{m,n}$ the portion of which within the angle is greater than $\frac{1}{2}$. Then

$$\overline{\lim_{|z_{m,n}|\to\infty}} \frac{\log|g(z_{m,n})|}{|z_{m,n}|^2} \geqslant \tfrac{1}{2}\pi, \tag{5.4}$$

because, if (5.4) were false, it would follow from (5.1), (5.2) by Lemma 6 that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r,g)}{r^2} \leqslant \tfrac{1}{2}\pi,$$

which contradicts (5.3). Hence (5.4) is true.

Thus there exists a sequence of squares $\{S_{m_s,n_s}\}$ ($s = 1, 2, 3,...$) tending to infinity, such that

$$\overline{\lim_{s\to\infty}} \frac{\log|g(z_{m_s,n_s})|}{|z_{m_s,n_s}|^2} \geqslant \tfrac{1}{2}\pi. \tag{5.5}$$

It follows from (5.1) and (5.2) that only a finite number of these squares meet the arms of the angle $\delta \leqslant \arg z \leqslant \frac{1}{2}\pi-\delta$. These may be omitted from the sequence $\{S_{m_s,n_s}\}$ which is renumbered accordingly. Let ζ_1^*, ζ_2^*, ζ_3^*,... be the centres of S_{m_1,n_1}, S_{m_2,n_2}, S_{m_3,n_3},..., respectively. Then the circle $|z-\zeta_s^*| \leqslant d$ lies within the closed square $S^*_{m_s,n_s}$, and it follows from the choice of z_{m_s,n_s} that, if z lies within this circle, then

$$|g(z)| \geqslant |g(z_{m_s,n_s})|.$$

Thus from (5.5), if ϵ is any positive number,

$$\frac{\log|g(z)|}{|z|^2} > (\tfrac{1}{2}\pi-\tfrac{1}{2}\epsilon),$$

provided that, for some ζ_s^* where $s > s_3(\epsilon)$, $|z-\zeta_s^*| < d$.

Let $\zeta_s = (\zeta_s^* e^{-i\pi/4}/\lambda)$. If $\epsilon > 0$,

$$\frac{\log|f(z)|}{|z|^2} > (\tfrac{1}{2}\pi-\tfrac{1}{2}\epsilon)\frac{1}{\lambda^2},$$

provided that $$\left|\frac{z}{\lambda}e^{\frac{1}{4}i\pi}-\zeta_s^*\right| < d \quad (s > s_3(\epsilon)),$$

that is $$|z-\zeta_s| < \lambda d \quad (s > s_3(\epsilon)).$$

Since λ may be chosen so that $1 < \lambda < \sqrt{\left(\frac{\pi-\epsilon}{\pi-2\epsilon}\right)}$, it follows that

$$\frac{\log|f(z)|}{|z|^2} > (\tfrac{1}{2}\pi-\epsilon),$$

provided that $|z-\zeta_s| < d$ ($< \lambda d$). This is the required result.

It is now possible to prove Theorem II.

Proof of Theorem II. Let

$$f_1(z) \equiv f\left[\left\{\sqrt{\left(\frac{\pi}{2\kappa}\right)}z\right\}^{2/\rho}\right].$$

Then
$$\overline{\lim}\frac{\log M(f_1(z), r, \alpha^*)}{r^2} = \tfrac{1}{2}\pi,$$

where
$$M\{f_1(z), r, \alpha^*\} = \max_{|\theta|\leqslant\alpha^*} |f_1(re^{i\theta})|,$$

and $\alpha^* = \frac{1}{2}\alpha\rho < \frac{1}{4}\pi$. Also

$$\overline{\lim_{r\to\infty}}\frac{\log|f_1(re^{\pm i\alpha^*})|}{r^2} < \tfrac{1}{2}\pi.$$

The function $f_1(z)$ is regular in the angle $|\arg z| \leqslant \alpha^* < \frac{1}{4}\pi$, and satisfies the further hypotheses of Lemma 7. Therefore, if d^* is such that $d < d^* < \frac{1}{2}$, there exists a sequence of points ζ_1^*, ζ_2^*, ζ_3^*,..., within the angle and distant at least d^* from its arms, such that

$$|\zeta_1^*| \leqslant |\zeta_2^*| \leqslant |\zeta_3^*| \leqslant ... \to \infty,$$

and that, given any $\epsilon > 0$,

$$\log|f_1(z)| > (\tfrac{1}{2}\pi - \epsilon)|z|^2,$$

provided that $|z-\zeta_s^*| < d^*$ and $s > s_3(\epsilon)$.

Hence

$$\log|f(z)| = \log\left|f_1\left\{\sqrt{\left(\frac{2\kappa}{\pi}\right)}z^{\frac{1}{2}\rho}\right\}\right| > (\tfrac{1}{2}\pi-\epsilon)\frac{2\kappa}{\pi}|z|^\rho = \left(\kappa - \frac{2\kappa\epsilon}{\pi}\right)|z|^\rho,$$

provided that $|\sqrt{(2\kappa/\pi)}z^{\frac{1}{2}\rho}-\zeta_s^*| < d^*$, $s > s_3(\epsilon)$. That is, provided that $|z^{\frac{1}{2}\rho}-\zeta_s^{\frac{1}{2}\rho}| < d^*\sqrt{(\pi/2\kappa)}$, where $\zeta_s = (\pi/2\kappa)^{1/\rho}\zeta_s^{*2/\rho}$, and $s > s_3(\epsilon)$.

But, if, as in the hypotheses of the theorem,

$$|z-\zeta_s| < (2d/\rho)\sqrt{(\pi/2\kappa)}|\zeta_s|^{1-\frac{1}{2}\rho},$$

it follows that $|z/\zeta_s| \to 1$ as $|\zeta_s| \to \infty$. Then

$$|(z^{\frac{1}{2}\rho}-\zeta_s^{\frac{1}{2}\rho})/(z-\zeta_s)| \to \tfrac{1}{2}\rho|\zeta_s|^{\frac{1}{2}\rho-1}.$$

The hypothesis on $|z-\zeta_s|$ given in the enunciation of the theorem leads, when $|\zeta_s|$ is sufficiently large, to

$$|z^{\frac{1}{2}\rho}-\zeta_s^{\frac{1}{2}\rho}| < \left(\frac{d^*}{d}\right)\tfrac{1}{2}\rho|z-\zeta_s||\zeta_s|^{\frac{1}{2}\rho-1}$$
$$< \frac{d^*}{d}\frac{2d}{\rho}\tfrac{1}{2}\rho\sqrt{\left(\frac{\pi}{2\kappa}\right)}$$

since $(d^*/d) > 1$. That is,

$$|z^{\frac{1}{2}\rho}-\zeta_s^{\frac{1}{2}\rho}| < d^*\sqrt{(\pi/2\kappa)}.$$

Hence $$\log|f(z)| > (\kappa-\epsilon)|z|^\rho,$$
provided that $|z-\zeta_s| < (2d/\rho)\sqrt{(\pi/2\kappa)}|\zeta_s|^{1-\frac{1}{2}\rho}$, $s > s_3\left(\epsilon\dfrac{2\kappa}{\pi}\right)$, and $|\zeta_s|$ is sufficiently large, that is, $s > s'(\epsilon)$, where $s' > s_3\left(\epsilon\dfrac{2\kappa}{\pi}\right)$.

6. Theorem II yields information about the flat regions of an integral function. Suppose that $f(z)$ is an integral function of order $\rho > 0$, and type $\kappa > 0$, and that
$$h(\theta) = \varlimsup_{r\to\infty}\frac{\log|f(re^{i\theta})|}{r^\rho}. \tag{6.1}$$
It is well known that $h(\theta)$ is a continuous function of θ, that $h(\theta) \leqslant \kappa$ for $0 \leqslant \theta < 2\pi$, and that $h(\theta)$ attains the value κ for some value or values of θ.

If $h(\theta)$ has a strict maximum at $\theta = \theta_0$, then there is an arbitrarily small $\delta > 0$ such that
$$\varlimsup_{r\to\infty}\frac{\log|f(re^{i(\theta_0\pm\delta)})|}{r^\rho} < h(\theta_0). \tag{6.2}$$
Hence, if $d < \frac{1}{2}$, $\epsilon > 0$, it follows from Theorem II that there exists a sequence $\{\zeta_s\}$, where $|\zeta_1| \leqslant |\zeta_2| \leqslant |\zeta_3| \leqslant ... \to \infty$, such that the circles
$$|z-\zeta_s| < \frac{2d}{\rho}\sqrt{\left(\frac{\pi}{2h(\theta_0)}\right)}|\zeta_s|^{1-\frac{1}{2}\rho}$$
lie in the angle $(\theta_0-\delta, \theta_0+\delta)$, and, if z is a point in the circle with centre ζ_s,
$$\log|f(z)| > \{h(\theta_0)-\epsilon\}|z|^\rho, \tag{6.3}$$
provided that $s > s''(\epsilon)$.

If $h(\theta_0) = \kappa$, but $h(\theta_0)$ is not a strict maximum, the problem needs further consideration. It is no real limitation to assume that $\theta_0 = 0$. Then, if $\epsilon_1 > 0$, let $g(z) \equiv f(z)\exp(\epsilon_1 z^\rho)$, and let
$$H(\theta) = \varlimsup_{r\to\infty}\frac{\log|g(re^{i\theta})|}{r^\rho}. \tag{6.4}$$
The function $H(\theta)$ has a maximum value $(\kappa+\epsilon_1)$ when $\theta = 0$. Let d be less than $\frac{1}{2}$, and ϵ_1 be so small that $\sqrt{\left(\dfrac{\kappa+\epsilon_1}{\kappa}\right)}d = d^* < \frac{1}{2}$. Then, if $\delta > 0$, $\epsilon > 0$, there exists a sequence of points $\{\zeta_s\}$, where $|\zeta_s|$ increases steadily with s and tends to infinity, such that
$$\log|g(z)| > (\kappa+\epsilon_1-\epsilon)|z|^\rho, \tag{6.5}$$

provided that

$$|z-\zeta_s| < \frac{2d^*}{\rho}\sqrt{\left(\frac{\pi}{2(\kappa+\epsilon_1)}\right)}|\zeta_s|^{1-\frac{1}{2}\rho} = \frac{2d}{\rho}\sqrt{\left(\frac{\pi}{2\kappa}\right)}|\zeta_s|^{1-\frac{1}{2}\rho},$$

and $s > s''(\epsilon)$. Also the above circles lie within the angle $|\arg z| < \delta$.

Since (6.5) is equivalent to

$$\log|f(z)| > (\kappa+\epsilon_1-\epsilon)|z|^\rho-\epsilon_1 Rz^\rho > (\kappa-\epsilon)|z|^\rho, \qquad (6.6)$$

Theorem III has been established.

In concluding this paper I should like to express my gratitude to Miss M. L. Cartwright for the suggestion that directed my attention to the behaviour of an integral function at points $\{z_{m,n}\}$, in place of the more usual lattice points $\{m+in\}$. This fruitful suggestion is the basis both of this and of my previous paper.

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